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# ON THE VARIANCE AND SKEWNESS OF THE SWAP RATE IN A STOCHASTIC VOLATILITY INTEREST RATE MODEL

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This paper provides new insight in the distribution of the (forward par) swap rate in a stochastic volatility model for the dynamics of the forward rate curve. First the swap rate dynamics are obtained in a multi-curve environment with deterministic spread. Then, the variance of the swap rate is derived making use of a result on the distribution of random variables generated by extended square-root diffusion processes. Also, the skewness is derived by Itô calculus. These results give rise to moment-matching swaption price formulas which are expected to permit a fast approximate calibration of the model.

*Keywords:* Interest rate models, Stochastic processes, Swaption pricing.

## 1. Introduction

Subsequent to the introduction of the Heston model (Heston, 1993) and especially since the early 2000's we have seen stochastic volatility models of increasing complexity to capture the apparently non-deterministic nature of interest rate volatility (Chen and Scott, 2001; Casassus et al., 2005; Andersen and Brotherton-Ratcliffe, 2005).

Clearly these models offer a more commensurate representation of the market than classical short-rate models (Casassus et al., 2005). Yet their practical use is complicated by the fact that elementary interest rates products (such as European swaptions) are only tractable by Fourier inversion methods and other numerical means which leads to a lengthy calibration process (Trolle and Schwartz, 2009). Similarly our understanding of the basic model-implied processes is hindered by the lack of closed-form expressions for their density functions at a given point of time (del Bano Rollin et al., 2010; Gulisashvili, 2012) and of results regarding the moments. Among these processes the one driving the evolution of the swap underlying, namely the (forward par) swap rate, is of special significance: the distribution of its values at option expiry (under the forward swap measure) directly determines the price of the European swaption. For example, as in (Grbac and Runggaldier, 2015, ch. 1.4.7), the value at time $t$ of a receiver swaption with strike $K$ and notional 1 becomes

$$V_{\text{rec}}(t) = A(t) \int_{-\infty}^{K} (K - x) f_X(x) dx, \tag{1}$$

where the random variable $X$ with density function $f_X(x)$ models the swap rate at option expiry and $A(t)$ denotes the annuity, a quantity that can directly be extracted from the yield curve, see (3).

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The purpose of this paper is to derive the variance and skewness of the distribution of swap rates in a concrete stochastic volatility interest rate model framework under so-called martingale-freezing (Brigo and Mercurio, 2006, ch. 6.15). These theoretical results give rise to approximative formulas for the swaption price by replacing the unknown $f_X(x)$ in (1) with a candidate density function $\widehat{f_X}(x)$ that matches (at least) the first 3 moments. Such expressions have been used to roughly calibrate the model to the market within seconds on a standard computer, in contrast to the computational effort required for pricing and calibrating with the usual numerical techniques (see appendix). Hence the proposed method is deemed to be of practical relevance as a tool for validation purposes or by providing a starting value for the actual calibration algorithm.

Before we set up the stochastic interest rate model framework, consider that in a standard interest rate swap a predetermined fixed rate is exchanged against a risky rate, whereas the rate implicit in the discount factors is assumed to be a, basically riskfree, Overnight Index Swap (OIS) rate, as usually prescribed in the Credit Support Annexes (CSAs) of secured transactions. Indeed, the OIS rate arises as the natural rate for discounting by being the accrual rate paid on posted cash collateral (Grbac and Runggaldier, 2015, ch. 1.3.1). The term "risky rate" in turn denotes any interest rate that reflects a certain degree of credit or liquidity risk, depending on the tenor, such as the LIBOR, EURIBOR or JIBAR (Grbac and Runggaldier, 2015, ch. 1.2.1). To model the combined dynamics of the risky rate and the OIS rate curve we set up a stochastic model for the latter and presume a time-dependent but deterministic spread between the two curves (see Section 2), as in Henrard (2009).

To define the model, note that given a curve of discount factors $T \mapsto P(t,T)$, where $T \geq t$, expressing the respective value at $t$ of the future payment of 1 unit at $T$, the instantaneous continuously compounded forward rate at $T$ observed at $t$ is defined as $f(t,T) := -\frac{\partial}{\partial T}\log(P(t,T))$, such that $P(t,T) = \exp(-\int_t^T f(t,s)ds)$. Thereby the forward rates are the appropriate discount rates applicable to outstanding cashflows implied by the market's view on their values today.

With this in mind, we assume that the dynamics of the OIS forward rates $T \mapsto f(t,T)$, where $T \geq t$, and their volatility process $\upsilon_t$ are driven by the model introduced in Trolle and Schwartz (2009)[1]:

$$\begin{aligned} df(t,T) &= \mu_f(t,T)dt + \sigma_f(t,T)\sqrt{\upsilon_t}dW_t, \\ d\upsilon_t &= \kappa(\theta - \upsilon_t)\,dt + \sigma\sqrt{\upsilon_t}\left(\rho dW_t + \sqrt{1-\rho^2}dZ_t\right), \end{aligned} \tag{2}$$

where $W_t$ and $Z_t$ are two independent Brownian motions under the risk-neutral measure. It is postulated that the diffusion term satisfies

$$\sigma_f(t,T) = (\alpha_0 + \alpha_1(T-t))\,e^{-\gamma(T-t)}$$

and, to assure the absence of arbitrage (Heath et al., 1992), we set[2]

$$\mu_f(t,T) = \upsilon_t\sigma_f(t,T)\int_t^T \sigma_f(t,u)du.$$

[1] Trolle and Schwartz (2009) and Trolle (2009) propose different ways to generalize this model from the one-dimensional setup, in their nomenclature $N = 1$, to arbitrary dimensions. To simplify matters and improve readability the whole discussion will assume $N = 1$, noting that Theorems 1 and 2 extend to $N > 1$ since the dimensions can in any case be treated separately.
[2] The expression for $\mu_f(t,T)$ is further spelled out in Trolle and Schwartz (2009, eq. (57)).

Thereby the model is determined by an 8-tuple of model parameters $(\alpha_0, \alpha_1, \gamma, \kappa, \theta, \sigma, \rho, \upsilon_0)$ which can be calibrated from swaption prices observed in the market[3]. For the CIR process we assume that $\kappa, \theta \geq 0$, $\sigma > 0$ as well as $\upsilon_0 > 0$. As in Trolle and Schwartz (2009), our derivations are valid for any real $\alpha_0$ and non-zero real $\alpha_1, \gamma$. For the correlation parameter we assume $\rho \in [-1, 1]$.

This volatility structure produces the empirically detected "humped" shape of forward rate volatility curves, initially rising and then falling monotonously along the bond maturity spectrum, (Ritchken and Chuang, 2000). Also, it is known that this framework guarantees the interest rate dynamics to be Markovian in augmented state space (Ritchken and Sankarasubramaniam, 1995). The model encompasses some classical short-rate models such as the Hull-White model and the Ho-Lee model as special cases. Also, if $\upsilon_t = \upsilon_0$ for all $t$, it specialises to the Mercurio and Moraleda model (Mercurio and Moraleda, 2000). Finally, the Trolle-Schwartz model fits in the general discussion of stochastic volatility models (compare e.g. Casassus et al., 2005).

The paper is organized as follows: In Section 2 we first formalize the construction of the interest rate swap starting from the mere observation of the OIS and risky rate curves in the market. Then we derive the corresponding stochastic dynamics of the swap rate. In Section 3 we use this result to arrive at the variance and skewness of the swap rate distribution. Section 4 concludes with possible applications and gives an outlook for further research.

## 2. The swap rate dynamics

In this section we will first explicate how we model an interest rate swap within our interest rate framework. Then we will provide the resulting dynamics of the swap rate in terms of stochastic differential equations (see Lemma 1). This result will be exploited in the next section in order to derive the variance and skewness of its distribution at the expiry of an option on this swap. The main technical tool in this section is Itô's formula which is presented in most introductory textbooks on stochastic analysis; see e.g. Karatzas and Shreve (1991, Theorem 3.3).

The starting point of this deduction is the OIS rate discount factor curve, $T \mapsto P(t,T)$, where $T \geq t$, seen from today: $t = 0$. Likewise we require the risky rate discount factor curve for a given tenor $x$, which corresponds to the duration of the interest rate period of the floating side of a swap (e.g. 6 months), $T \mapsto P^x(t,T)$, $T \geq t$, at $t = 0$. Both of them may have been constructed from the market prices of interest rate products (e.g. futures, forward rate agreements, swaps), a daily exercise in a bank (Henrard, 2009, ch. 3).

Recall that we intend to model the evolution of the OIS discount factors $P(t,T)$ by assuming a Trolle-Schwartz model for the forward rates $f(t,T) = -\frac{\partial}{\partial T}\log(P(t,T))$, see (2), and for the resulting dynamics of $P(t,T)$, see Trolle and Schwartz (2009, eq. (20)). To establish the connection between the two curves we define the product of discount factor ratios

$$\beta_t^x(S,T) := \frac{P^x(t,S)}{P^x(t,T)}\frac{P(t,T)}{P(t,S)},$$

where $0 \leq t \leq S \leq T$. As in Henrard (2009, hypothesis S0) we assume that, given $S$ and $T$, $\beta_t^x(S,T) = \beta_0^x(S,T)$ for all $t$, i.e. the term is constant through time. It follows that for every future

[3]Note that among $(\alpha_0, \alpha_1, \theta, \sigma, \upsilon_0)$ one parameter can be chosen at will (which corresponds to rescaling $\upsilon_t$ by a constant positive factor), i.e. there are in fact 7 degrees of freedom instead of 8.

$t \geq 0$ the spread curve $T \mapsto \lambda^x(t,T)$, where $T \geq t$, defined by $P^x(t,T) = P(t,T)e^{-\lambda^x(t,T)(T-t)}$, is already determined at $t = 0$.[4]

Let us now move on to the derivation of the swap rate dynamics according to this setup: In a standard interest rate swap a preset fixed rate is exchanged against a floating rate, determined at the start of each period and paid at the end. The (forward par) swap rate is then defined to be the appropriate fixed rate such that the value of the swap equals 0.

Historically (before the financial crisis of 2007–08), both the discount factors and the floating rate were derived from the same curve (Grbac and Runggaldier, 2015, ch. 1). Say, for a swap running from $T_m$ to $T_n$, with variable coupons of size $[P(T_j,T_{j+1})]^{-1} - 1$, $j = m,\dots,n-1$, fixed at $T_j$ and paid at $T_{j+1}$ (the end of the period), as well as $\tau_j := T_{j+1} - T_j$, the single-curve swap rate observed at $t \leq T_m$ becomes[5]

$$S_t(T_m,T_n) = \frac{\sum_{j=m}^{n-1} \tau_j P(t,T_{j+1}) L(t,T_j,T_{j+1})}{A(t)} = \frac{P(t,T_m) - P(t,T_n)}{A(t)},$$

where $L(t,T_j,T_{j+1}) := \tau_j^{-1}[P(t,T_j)/P(t,T_{j+1}) - 1]$ is the discretely compounded forward rate for the time interval $[T_j,T_{j+1}]$ and

$$A(t) := \sum_{j=m}^{n-1} \tau_j P(t,T_{j+1}) \tag{3}$$

denotes the annuity; see also Brigo and Mercurio (2006, ch. 1.5).

In our multi-curve setup the risk-free forward rate $L(t,T_j,T_{j+1})$ is replaced by

$$L^x(t,T_j,T_{j+1}) := \frac{1}{\tau_j}\left(\frac{P^x(t,T_j)}{P^x(t,T_{j+1})} - 1\right) = \frac{1}{\tau_j}\left(\frac{P(t,T_j)}{P(t,T_{j+1})}\beta_j^x - 1\right),$$

with $\beta_j^x := \beta_t^x(T_j,T_{j+1})$, which by assumption does in fact not depend on $t$. The swap rate for a swap against variable coupons of size $[P^x(T_j,T_{j+1})]^{-1} - 1$ with equal payment schedule as above becomes

$$S_t^x(T_m,T_n) = \frac{\sum_{j=m}^{n-1} \tau_j P(t,T_{j+1}) L^x(t,T_j,T_{j+1})}{A(t)}. \tag{4}$$

To disentangle the payoff and discounting components of the swaption price it is convenient to consider the stochastic model under the forward swap measure $\mathbb{P}_0^{m,n}$ (Privault, 2014, ch. 12), which is defined by its Radon-Nikodym derivative relative to the risk-neutral measure $\mathbb{P}^*$:

$$\frac{d\mathbb{P}_0^{m,n}}{d\mathbb{P}^*} = e^{-\int_0^{T_m} r_s ds} \frac{A(T_m)}{A(0)},$$

[4] Indeed, taking $S = t$ we get $\beta_t^x(t,T) = P(t,T)/P^x(t,T) = e^{\lambda^x(t,T)(T-t)}$. By assumption this equals

$$\beta_0^x(t,T) = \frac{P^x(0,t)}{P^x(0,T)} \frac{P(0,T)}{P(0,t)} = \frac{e^{-\lambda^x(0,t)t}}{e^{-\lambda^x(0,T)T}}.$$

After rearraging we obtain $\lambda^x(t,T) = [\lambda^x(0,T)T - \lambda^x(0,t)t]/(T-t)$, which can be observed today.

[5] $S_t(T_m,T_n)$ is simply the solution for the fixed rate $K$ in $A(t)K = \sum_{j=m}^{n-1} \tau_j P(t,T_{j+1}) L(t,T_j,T_{j+1})$, where the values of the two opposing legs are set equal for a swap value of 0.

with the short-rate process $r_s := f(s,s)$.[6]

In the next lemma we write the dynamics of $S_t^x(T_m,T_n)$ as a stochastic differential equation driven by a Brownian motion under this changed measure.

**Lemma 1.** *Denote by $S_t^x(T_m,T_n)$ the swap rate of a swap against the risky rate, running from $T_m$ to $T_n$, with floating payments fixed at $T_j$ and paid at $T_{j+1} = T_j + \tau_j$, $j = m, ..., n-1$, as before.*

*For $\tau \geq 0$, let $B_x(\tau) = \alpha_1\gamma^{-1}[(\gamma^{-1} + \alpha_0\alpha_1^{-1})(e^{-\gamma\tau} - 1) + \tau e^{-\gamma\tau}]$. Then $S_t^x(T_m,T_n)$ evolves according to*

$$dS_t^x(T_m,T_n) = \sum_{j=m}^{n} \zeta_j^x(t) B_x(T_j - t)\sqrt{v_t}\, dW_t^{m,n}, \tag{5}$$

*where $W_t^{m,n}$ denotes a Brownian motion under the forward swap measure and*

$$\begin{aligned}
\zeta_m^x(t) &= \frac{P(t,T_m)}{A(t)}\beta_m^x,\\
\zeta_j^x(t) &= \frac{P(t,T_j)}{A(t)}\Big(\beta_j^x - 1 - \tau_{j-1}S_t^x(T_m,T_n)\Big), \quad j = m+1, ..., n-1,\\
\zeta_n^x(t) &= -(1 + \tau_{n-1}S_t^x(T_m,T_n))\frac{P(t,T_n)}{A(t)}.
\end{aligned}$$

*Proof.* By Itô's formula we can write

$$dS_t^x(T_m,T_n) = \sum_{j=m}^{n-1} \frac{\partial S_t^x(T_m,T_n)}{\partial L(t,T_j,T_{j+1})} dL(t,T_j,T_{j+1}) + (...)dt, \tag{6}$$

i.e. up to additional drift terms. Observe that there are no other random variables involved, such that the Brownian motion part of the differential, contained in the $dL(t,T_j,T_{j+1})$, is complete. Furthermore, $A(t)S_t^x(T_m,T_n)$, equating to a swap with a fixed rate of 0, is a tradable product, such that $S_t^x(T_m,T_n)$ is a martingale under the forward swap measure (Privault, 2014, ch. 12.4), and hence we already know that the overall drift term of $dS_t^x(T_m,T_n)$ will vanish. Straightforward differentiation of the sum of products

$$\frac{\partial S_t^x(T_m,T_n)}{\partial L(t,T_j,T_{j+1})} = \frac{\partial}{\partial L(t,T_j,T_{j+1})}\left\{\sum_{k=m}^{n-1} \frac{\tau_k P(t,T_{k+1})}{A(t)} L^x(t,T_k,T_{k+1})\right\}$$

[6] With regards to the risk-neutral measure the price of, for example, a receiver swaption with strike $K$ and notional 1 becomes $V_{rec}(t) = \mathrm{E}^*[\exp(-\int_t^{T_m} r_s ds)(\sum_{j=m}^{n-1} P(T_m,T_{j+1})(K - L^x(T_m,T_j,T_{j+1})))^+|\mathcal{F}_t]$, where the $\sigma$-algebra $\mathcal{F}_t$ contains the information available at $t$. After a change of numéraire we can rewrite this as an expected value under the forward swap measure, $V_{rec}(t) = A(t)\mathrm{E}_0^{m,n}[(K - S_{T_m}^x(T_m,T_n))^+ \mid \mathcal{F}_t]$, as in Privault (2014, ch. 12), which is simply (1) with $X = S_{T_m}^x(T_m,T_n)|\mathcal{F}_t$.

as in Wu (2019, Lemma 6.4.1) yields

$$\frac{\partial S_t^x(T_m,T_n)}{\partial L(t,T_j,T_{j+1})}$$

$$= \sum_{k=m}^{n-1} \frac{\tau_k P(t,T_{k+1})}{A(t)} \frac{\partial L^x(t,T_k,T_{k+1})}{\partial L(t,T_j,T_{j+1})} + \sum_{k=m}^{j-1} \frac{\partial \left\{ \frac{\tau_k P(t,T_{k+1})}{A(t)} \right\}}{\partial L(t,T_j,T_{j+1})} L^x(t,T_k,T_{k+1}) \tag{7}$$

$$= \frac{\tau_j P(t,T_{j+1})}{A(t)} \beta_j^x + \frac{\tau_j}{1+\tau_j L(t,T_j,T_{j+1})} \sum_{k=m}^{j-1} \frac{\tau_k P(t,T_{k+1})}{A(t)} \left( L^x(t,T_k,T_{k+1}) - S_t^x(T_m,T_n) \right),$$

using

$$\frac{\partial L^x(t,T_k,T_{k+1})}{\partial L(t,T_j,T_{j+1})} = \frac{\partial}{\partial L(t,T_j,T_{j+1})} \left\{ \beta_k^x L(t,T_k,T_{k+1}) + \frac{1}{\tau_k}(\beta_k^x - 1) \right\} = \beta_j^x \mathbb{1}_{j=k}.$$

Regarding the other factors in (6), $dL(t,T_j,T_{j+1}) = \tau_j^{-1} d(P(t,T_j)/P(t,T_{j+1}))$, for $j = m, ..., n-1$, we know that

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + B_x(T-t)\sqrt{\upsilon_t} dW_t,$$

for any $T \geq t$ (Trolle and Schwartz, 2009, eq. 28), with $W_t$ a Brownian motion under the risk-neutral measure. By making use of $dW_t^{m,n} = dW_t + (...)dt$ (Trolle, 2009, eq. 61), i.e. the diffusion term is not altered when moving to the forward swap measure, another application of Itô's formula yields

$$dL(t,T_j,T_{j+1}) = \left\{ \sqrt{\upsilon_t} \frac{1}{\tau_j} \left(1 + \tau_j L(t,T_j,T_{j+1})\right) \left(B_x(T_j - t) - B_x(T_{j+1} - t)\right) \right\} dW_t^{m,n} + (...)dt. \tag{8}$$

Multiplying (8) and the expression obtained last in (7), summing over $j$ and grouping terms by $B_x(T_j - t)$ we obtain the result. ∎

Note that Lemma 1 specializes to the case $N = 1$ of equations (62), (63) in Trolle (2009) when $\beta_j^x = 1$ for all $j$, i.e. in the single-curve setting.

Finally we cite the corresponding result for the process $\upsilon_t$ to have a full description of the model at our disposal, suitable for the derivation of some of its statistical properties in the next section. According to Trolle (2009, eq. (64)),

$$d\upsilon_t = \left( \kappa(\theta - \upsilon_t) + \upsilon_t \sigma \rho \sum_{j=m+1}^{n} \xi_j(t) B_x(T_j - t) \right) dt + \sigma\sqrt{\upsilon_t} \left( \rho dW_t^{m,n} + \sqrt{1-\rho^2} dZ_t^{m,n} \right), \tag{9}$$

with $B_x(\tau)$ as in Lemma 1, $\xi_j(t) = \tau_{j-1} P(t,T_j)/A(t)$ and where $Z_t^{m,n}$ is another Brownian motion under the forward swap measure, independent of $W_t^{m,n}$.

## 3. Variance and skewness of the swap rate

First of all, let us briefly recall that in (4) we defined the swap rate $S_t^x(T_m,T_n)$ observed at $t$ for an interest rate swap, running from $T_m$ to $T_n$, with variable coupons of size $[P^x(T_j,T_{j+1})]^{-1} - 1$, fixed at $T_j$ and paid at $T_{j+1} = T_j + \tau_j$, where $j = m, ..., n-1$. The section concluded with its dynamics under the forward swap measure $\mathbb{P}_0^{m,n}$ which are specified by equations (5) and (9).

Next we use these results to get the variance and skewness of the random variable $S_t^x := S_t^x(T_m, T_n)$ at the option expiry $t = T_m$ under this measure, given the model parameters $(\alpha_0, \alpha_1, \gamma, \kappa, \theta, \sigma, \rho, \upsilon_0)$. Prior to this we note that, being a driftless diffusion process, $S_t^x$ is especially a martingale, such that its starting value is also the expected value for a future time: $\mathrm{E}[S_{T_m}^x] = S_0^x$, introducing the shorthand $\mathrm{E}[\cdot] := \mathrm{E}_0^{m,n}[\cdot]$.

In what follows, the martingales $\zeta_j^x(t)$ and $\xi_j(t)$ in (5) and (9) are replaced by their values at $t = 0$ ("freezing technique"). Since their enumerator and denominator are expected to move in line, their variance is typically regarded as negligible (see e.g. Schrager and Pelsser, 2006).

Under this assumption we will obtain the variance of $S_{T_m}^x$ by means of the classical Itô isometry (Karatzas and Shreve, 1991, eq. 2.14) and by making use of the distributional properties of the extended CIR (Cox-Ingersoll-Ross) process $\upsilon_t$ appearing in (5). To arrive at the skewness in turn we will generalize a method introduced for the Heston model in Zhang et al. (2017), which relies mainly on Itô's formula.

The following theorem provides the variance of $S_{T_m}^x$.

**Theorem 1.** *The second moment of $S_{T_m}^x$ is given by*

$$\mathrm{Var}\left(S_{T_m}^x\right) = \int_0^{T_m} \left(\upsilon_0 e^{b_2} - \frac{d}{4}\sigma^2 \sum_{l=0}^{\infty} \frac{a_2^l}{g(l)^{l+1}} \{e^{-g(l)t} J(t,l) - J(0,l)\}\right) \times q(t) \left(\sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - t)\right)^2 dt,$$

*where*

$$a_2 = \frac{a}{\gamma}, \quad a = \sigma\rho\frac{\alpha_1}{\gamma} \sum_{j=m+1}^{n} \xi_j(0) e^{-\gamma T_j}, \quad b_2 = \frac{b}{\gamma} - \frac{a}{\gamma^2},$$

$$b = -\sigma\rho\frac{\alpha_1}{\gamma} \sum_{j=m+1}^{n} \xi_j(0) e^{-\gamma T_j} \left(\frac{1}{\gamma} + \frac{\alpha_0}{\alpha_1} + T_j\right), \quad d = \frac{4\kappa\theta}{\sigma^2},$$

$$g(l) = -l\gamma - c, \quad c = \sigma\rho\frac{\alpha_1}{\gamma}\left(\frac{1}{\gamma} + \frac{\alpha_0}{\alpha_1}\right) \sum_{j=m+1}^{n} \xi_j(0) + \kappa,$$

*and with certain elementary functions $q(t)$ and $J(x,l)$ defined in (15) and (18) in the proof.*

*Proof.* According to the Itô isometry the variance of $S_{T_m}^x$ can be written as

$$\mathrm{Var}\left(S_{T_m}^x\right) = \int_0^{T_m} \left(\sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - t)\right)^2 \mathrm{E}[\upsilon_t] dt. \tag{10}$$

The expected value can be made explicit as follows: Observe that

$$d\upsilon_t = (\kappa\theta - b(t)\upsilon_t)\, dt + \sigma\sqrt{\upsilon_t}\left(\rho dW_t^{m,n} + \sqrt{1-\rho^2} dZ_t^{m,n}\right), \tag{11}$$

where $b(t) := \kappa - \sigma\rho\sum_{j=m+1}^{n}\xi_j(0)B_x(T_j - t)$. For this extended CIR process Maghsoodi (1996) has shown that, up to a factor, $\upsilon_t$ is noncentral chi-square distributed. Concretely,

$$\Sigma(0,t)^{-1}\upsilon_t \sim \chi^2\left(d, \frac{\upsilon_0 e^{-\int_0^t b(u)du}}{\Sigma(0,t)}\right),$$

where $\Sigma(0,t) := \frac{1}{4}\sigma^2\int_0^t \exp(-\int_s^t b(u)du)ds$ and $d := 4\kappa\theta/\sigma^2$. According to Johnson et al. (1995, ch. 29) it follows that

$$\mathrm{E}[\upsilon_t] = \upsilon_0 e^{-\int_0^t b(u)du} + d\Sigma(0,t). \tag{12}$$

To spell out the integral term $\Sigma(0,t)$, let us write

$$b(t) = e^{\gamma t}(at+b) + c, \tag{13}$$

with

$$a := \sigma\rho\frac{\alpha_1}{\gamma}\sum_{j=m+1}^{n}\xi_j(0)e^{-\gamma T_j}, \quad b := -\sigma\rho\frac{\alpha_1}{\gamma}\sum_{j=m+1}^{n}\xi_j(0)e^{-\gamma T_j}\left(\frac{1}{\gamma}+\frac{\alpha_0}{\alpha_1}+T_j\right),$$
$$c := \sigma\rho\frac{\alpha_1}{\gamma}\left(\frac{1}{\gamma}+\frac{\alpha_0}{\alpha_1}\right)\sum_{j=m+1}^{n}\xi_j(0) + \kappa.$$

We have

$$e^{-\int_s^t b(u)du} = q(t)e^{e^{\gamma s}(a_2 s + b_2)+cs}, \tag{14}$$

with $a_2 := \frac{a}{\gamma}$, $b_2 := \frac{b}{\gamma} - \frac{a}{\gamma^2}$ and

$$q(t) := e^{-\left(e^{\gamma t}(a_2 t + b_2)+ct\right)}. \tag{15}$$

We obtain the following form whose integrand we develop into a series:

$$\Sigma(0,t) = \frac{1}{4}\sigma^2 q(t)\int_0^t e^{e^{\gamma s}(a_2 s+b_2)+cs}ds = \frac{1}{4}\sigma^2 q(t)\lim_{L\to\infty}\int_0^t\sum_{l=0}^{L}\frac{e^{cs}e^{l\gamma s}(a_2 s+b_2)^l}{l!}ds. \tag{16}$$

Now we use Gradshteyn et al. (2007, eq. 2.33.11) to get the following expression:

$$\Sigma(0,t) = \frac{1}{4}\sigma^2 q(t)\lim_{L\to\infty}\sum_{l=0}^{L} -\frac{a_2^l}{g(l)^{l+1}}\{e^{-g(l)t}J(t,l) - J(0,l)\}, \tag{17}$$

where $g(l) := -l\gamma - c$ and

$$J(x,l) := \sum_{k=0}^{l}\frac{\left(\frac{a_2 x+b_2}{a_2}g(l)\right)^k}{k!}. \tag{18}$$

Finally, using (17) and (14) with $s = 0$, (12) is plugged into (10). ■

Using parameters observed in practice, the limit is well approximated by developing the sum in (17) up to e.g. $L = 10$. The variance can then be determined by numerical integration.

Interestingly, replacing $\upsilon_t$ by its expected value as in Trolle and Schwartz (2009) leads to the same variance, but, as the diffusion term in the dynamics of $dS_t^x$ becomes deterministic, this enforces a normal distribution of $S_{T_m}^x$ and thereby restrains the higher moments.

Originally a main motivation to introduce stochastic volatility models was to be able to reproduce the skewed curve of implied swaption volatilities observed in the market (Brigo and Mercurio, 2006, ch. 11). In the next theorem we will derive the skewness of $S_{T_m}^x$ as a function of the model parameters. The subsequent remark recovers the fact that this skewness essentially arises from the correlation between the Brownian motions driving the processes $S_t^x$ and $\upsilon_t$.

**Theorem 2.** *Let $\rho \neq 0$. Then the third moment of $S_{T_m}^x$ equals*

$$\begin{aligned}
\mu_3\left(S_{T_m}^x\right) &= E\left[\left(S_{T_m}^x - E[S_{T_m}^x]\right)^3\right] \\
&= 3\left(\frac{a_3}{a_2}\right)^2 \int_0^{T_m} \sum_{l=0}^{\infty} (-1)^{l+1} a_2^l \{e^{-g_2(l)T_m} K(T_m, l) - e^{-g_2(l)t} K(t,l)\} \\
&\qquad \times \sigma\rho E[\upsilon_t] q(t)^{-1} \sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - t) dt,
\end{aligned}$$

*where*

$$\begin{aligned}
&a_3 = -\frac{\alpha_1}{\gamma} \sum_{j=m}^{n} \zeta_j^x(0) e^{-\gamma T_j}, \quad a_2 = \frac{a}{\gamma}, \quad a = \sigma\rho\frac{\alpha_1}{\gamma} \sum_{j=m+1}^{n} \xi_j(0) e^{-\gamma T_j}, \\
&b_2 = \frac{b}{\gamma} - \frac{a}{\gamma^2}, \quad b = -\sigma\rho\frac{\alpha_1}{\gamma} \sum_{j=m+1}^{n} \xi_j(0) e^{-\gamma T_j}\left(\frac{1}{\gamma} + \frac{\alpha_0}{\alpha_1} + T_j\right), \quad d = \frac{4\kappa\theta}{\sigma^2}, \\
&g_2(l) = -l\gamma + c, \quad c = \sigma\rho\frac{\alpha_1}{\gamma}\left(\frac{1}{\gamma} + \frac{\alpha_0}{\alpha_1}\right) \sum_{j=m+1}^{n} \xi_j(0) + \kappa,
\end{aligned}$$

*$E[\upsilon_t]$ from (12), and with certain elementary functions $q(t)$ and $K(x,l)$ defined respectively in (15) above and in (25) in the proof.*

*Proof.* The proof follows similar arguments as in Zhang et al. (2017). To simplify notation, denote $\tilde{S}_t := S_t^x - \mathrm{E}[S_{T_m}^x] = S_t^x - S_0^x$.

Then by Itô's formula,

$$\begin{aligned}
\mathrm{E}[\tilde{S}_{T_m}^3] &= \mathrm{E}\int_0^{T_m} d\left(\tilde{S}_t^3\right) = \mathrm{E}\int_0^{T_m} 3\tilde{S}_t^2 d\tilde{S}_t + 3\tilde{S}_t d[\tilde{S}]_t \\
&= 3\int_0^{T_m} \left(\sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - t)\right)^2 \mathrm{E}[\upsilon_t \tilde{S}_t] dt.
\end{aligned}$$

By the product rule of stochastic integration,

$$\begin{aligned}
\mathrm{E}[\upsilon_t \tilde{S}_t] &= \mathrm{E}\int_0^t d(\upsilon_u \tilde{S}_u) = \mathrm{E}\int_0^t \upsilon_u d\tilde{S}_u + \tilde{S}_u d\upsilon_u + d[\tilde{S}, \upsilon]_u \\
&= \mathrm{E}\int_0^t \tilde{S}_u(\kappa\theta - b(u)\upsilon_u) + \sigma\rho \sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - u)\upsilon_u du \\
&= -\int_0^t b(u)\mathrm{E}[\upsilon_u \tilde{S}_u] du + \sigma\rho \int_0^t \sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - u)\mathrm{E}[\upsilon_u] du,
\end{aligned} \tag{19}$$

with $b(u) = e^{\gamma u}(au + b) + c$ as in (13) in the dynamics (11).

Taking the derivative $\frac{d}{dt}$ on both sides of (19) yields an inhomogenous linear differential equation of first order with variable coefficients for $f(t) := \mathrm{E}[\upsilon_t \tilde{S}_t]$:

$$f'(t) = -b(t) f(t) + f_0(t), \tag{20}$$

where

$$f_0(t) := \sigma\rho \sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - t)\mathrm{E}[\upsilon_t]. \tag{21}$$

The solution with initial value $f(0) = 0$ is

$$f(t) = e^{-\int_0^t b(u)du} \int_0^t e^{\int_0^u b(s)ds} f_0(u) du.$$

It follows that

$$\mathrm{E}[\tilde{S}_{T_m}^3] = 3\int_0^{T_m} h(t) \int_0^t e^{\int_0^u b(s)ds} f_0(u) du dt,$$

with $h(t) := (\sum_{j=m}^n \zeta_j^x(0) B_x(T_j - t))^2 \exp(-\int_0^t b(u)du)$. Let $H(t)$ denote an antiderivative of the latter, i.e. $H'(t) = h(t)$. Then, by the usual integration by parts,

$$\mathrm{E}[\tilde{S}_{T_m}^3] = 3\int_0^{T_m} (H(T_m) - H(t))\, e^{\int_0^t b(u)du} f_0(t) dt. \tag{22}$$

We show how such an $H(t)$ can be obtained. First write

$$\sum_{j=m}^{n} \zeta_j^x(0) B_x(T_j - t) = e^{\gamma t}(a_3 t + b_3) + c_3,$$

with the parameters

$$\begin{aligned}
a_3 &:= -\frac{\alpha_1}{\gamma} \sum_{j=m}^{n} \zeta_j^x(0) e^{-\gamma T_j}, \\
b_3 &:= \frac{\alpha_1}{\gamma} \sum_{j=m}^{n} \zeta_j^x(0) e^{-\gamma T_j} \left(\frac{1}{\gamma} + \frac{\alpha_0}{\alpha_1} + T_j\right), \\
c_3 &:= -\frac{\alpha_1}{\gamma} \left(\frac{1}{\gamma} + \frac{\alpha_0}{\alpha_1}\right) \sum_{j=m}^{n} \zeta_j^x(0).
\end{aligned}$$

Then, as in (14),

$$e^{-\int_0^t b(u)du} = q(t)e^{b_2}, \tag{23}$$

with $q(t) = \exp(-(e^{\gamma t}(a_2 t + b_2) + ct))$, $a_2 = a\gamma^{-1}$ and $b_2 = b\gamma^{-1} - a\gamma^{-2}$. Writing $q(t)$ as a series, we may state

$$H(t) = e^{b_2} \lim_{L\to\infty} \int_0^t (e^{\gamma s}(a_3 s + b_3) + c_3)^2 \sum_{l=0}^{L} (-1)^l \frac{e^{(\gamma l - c)s}(a_2 s + b_2)^l}{l!} ds.$$

An equivalent expression that can be treated as (16) reads

$$H(t) = e^{b_2} \left(\frac{a_3}{a_2}\right)^2 \lim_{L\to\infty} \sum_{l=0}^{L} \int_0^t (e^{\gamma s}(a_2 s + b_2) + e^{\gamma s} b_4 + c_4)^2 (-1)^l \frac{e^{(\gamma l - c)s}(a_2 s + b_2)^l}{l!} ds,$$

with $b_4 := -b_2 + b_3 a_2 a_3^{-1}$, $c_4 := c_3 a_2 a_3^{-1}$. For each $l$ expanding the squared brackets under the integral yields 6 integrals that can again all be transformed according to Gradshteyn et al. (2007, eq. 2.33.11). The terms can then be grouped as follows:

$$H(t) = e^{b_2} \left(\frac{a_3}{a_2}\right)^2 \lim_{L\to\infty} \sum_{l=0}^{L} (-1)^{l+1} a_2^l \{e^{t(\gamma l - c)} K(t,l) - K(0,l)\}, \tag{24}$$

with

$$\begin{aligned} K(x,l) := {} & \frac{\left(\frac{(l+1)(l+2)a_2^2}{g_2(l+2)^2} + \frac{2(l+1)a_2 b_4}{g_2(l+2)} + b_4^2\right) e^{2\gamma x}}{g_2(l+2)^{l+1}} \sum_{k=0}^{l} \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l+2)\right)^k}{k!} \\ & + \frac{\left(\frac{2(l+1)a_2 c_4}{g_2(l+1)} + 2b_4 c_4\right) e^{\gamma x}}{g_2(l+1)^{l+1}} \sum_{k=0}^{l} \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l+1)\right)^k}{k!} \\ & + \frac{c_4^2}{g_2(l)^{l+1}} \sum_{k=0}^{l} \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l)\right)^k}{k!} + M(x,l), \end{aligned} \tag{25}$$

with the function $g_2(l) := -\gamma l + c$ and residual terms

$$\begin{aligned} M(x,l) := {} & \frac{(l+1)(l+2)a_2^2 e^{2\gamma x}}{g_2(l+2)^{l+3}} \left( \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l+2)\right)^{l+1}}{(l+1)!} + \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l+2)\right)^{l+2}}{(l+2)!} \right) \\ & + \frac{2(l+1)a_2 b_4 e^{2\gamma x}}{g_2(l+2)^{l+2}} \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l+2)\right)^{l+1}}{(l+1)!} + \frac{2(l+1)a_2 c_4 e^{\gamma x}}{g_2(l+1)^{l+2}} \frac{\left(\frac{a_2 x + b_2}{a_2} g_2(l+1)\right)^{l+1}}{(l+1)!}. \end{aligned}$$

Plugging in (24), the inverse of (23) as well as (21) with the expected value $\mathrm{E}_0^{m,n}[v_t]$ from (12) into (22) a numerical integration yields $\mathrm{E}[\tilde{S}_{T_m}^3] = \mu_3$. ■

**Remark 1.** For $\rho = 0$ the skewness vanishes. Indeed, under this condition in (20) we have $f_0(t) = 0$ and $b(t) = \kappa$ for all $t$, such that $f(0) = 0$ implies $f(t) = 0$ for all $t$.

## 4. Summary and outlook

The preceding sections dealt with the dynamics of the swap rate in a stochastic volatility interest rate model in a multi-curve setting. First the relevant stochastic differential equations have been provided. Then the variance and skewness of the swap rate have been derived. Along with the expected value, which under the assumed forward swap measure is simply the starting value, this means that the first three moments that characterize the distribution of the swap rate are now known in explicit form.

As indicated in the introduction, these results give rise to approximate moment-matching formulas for the swaption price. More explicitly, let the swap rate be represented by a random variable $X = S^x_{T_m}(T_m, T_n)$ with density function $f_X(x)$. Then recall from (1) that the price of a receiver swaption with strike $K$ and notional 1 at time $t = 0$ equals

$$V_{\text{rec}}(0) = A(0)\mathrm{E}_0^{m,n}\left[(K - S^x_{T_m}(T_m, T_n))^+\right] = A(0)\int_{-\infty}^{K}(K - x) f_X(x)dx.$$

Having the expected value, variance and skewness of $X$ at hand we can replace the unknown $f_X(x)$ with some other density function $\widehat{f_X}(x)$ whose parameters have been chosen to match these moments. In this regard, three-parameter families such as skewed logistic (Johnson et al., 1995, ch. 23.10) or skewed hyperbolic secant (Cook, 2016) that can account for skewness and provide excess kurtosis are potential candidates.

Given the observed ease and speed of the implementation of the moment formulas in Theorems 1 and 2, they provide a means for a fast rough calibration procedure that circumvents the computational effort when using the actual model price (see Appendix). It is envisaged to analyze this application further in another research work.

## Appendix

To complete the presentation of swaptions in the multi-curve Trolle-Schwartz model, this appendix outlines the implementation of the pricing formulas for the payer and receiver swaption, $V_{\text{pay}}(t)$ and $V_{\text{rec}}(t)$.

The model price of a swaption is in fact also an approximation that deals with the problem of a multitude of dependent discount factors by replacing the underlying swap with a zero-coupon bond with equal volatility.

For convenience we repeat the result regarding the price of a bond option stated in Trolle and Schwartz (2009) and then describe how the extension to swaptions proposed in the appendix of Trolle and Schwartz (2009) can be adopted to a multi-curve setting.

The authors demonstrate that the price $P(t, T_0, T_1, K)$ at time $t$ of a put option on $P(t, T_1)$, i.e. on a zero-coupon bond with maturity $T_1$, having strike price $K$ and expiry $T_0$ can be traced back to the transform

$$\psi(u, t, T_0, T_1) := \mathrm{E}^*\left[e^{-\int_t^{T_0} r_s ds} e^{u\log(P(T_0,T_1))}\middle|\mathcal{F}_t\right]$$

as follows:

$$P(t, T_0, T_1, K) = KG_{0,1}(\log(K)) - G_{1,1}(\log(K)),$$

where

$$G_{a,b}(y) := \frac{\psi(a, t, T_0, T_1)}{2} - \frac{1}{\pi}\int_0^{\infty}\frac{\Im(\psi(a + iub, t, T_0, T_1)e^{-iuy})}{u}du \tag{26}$$

and $\Im(z)$ denotes the imaginary part of a complex number $z$.

According to Trolle and Schwartz (2009, Prop. 2) we have

$$\psi(u,t,T_0,T_1) = e^{M(T_0-t)+N(T_0-t)v_t+u\log(P(t,T_1))+(1-u)\log(P(t,T_0))},$$

where $M(\tau), N(\tau)$ solve the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dM(\tau)}{d\tau} &= N(\tau)\kappa\theta, \\
\frac{dN(\tau)}{d\tau} &= N(\tau)\left(-\kappa + \sigma\rho(uB_x(T_1 - T_0 + \tau) + (1-u)B_x(\tau))\right) + \frac{1}{2}N(\tau)^2\sigma^2 \\
&\quad + \frac{1}{2}(u^2 - u)B_x(T_1 - T_0 + \tau)^2 + \frac{1}{2}\left((1-u)^2 - (1-u)\right)B_x(\tau)^2 \\
&\quad + u(1-u)B_x(T_1 - T_0 + \tau)B_x(\tau),
\end{aligned}$$

with boundary conditions $M(0) = 0$ and $N(0) = 0$.

The functions $N(\tau)$ and $M(\tau)$ have to be implemented by numerical means, such as Runge-Kutta methods. Observe that this problem has to be solved repeatedly for each evaluation of the integrand in (26).

Recalling the notation in Section 2, let us now move on to a swaption on a swap running from $T_m$ to $T_n$, with floating payments of size $[P^x(T_j, T_{j+1})]^{-1} - 1$ fixed at $T_j$ and paid at $T_{j+1} = T_j + \tau_j$, $j = m, \ldots, n-1$. To see how the actual swap underlying can be traced back to a bond, observe that at expiry the value of a payer swaption equals

$$V_{\text{pay}}(T_m) = \left(\beta_m^x + \sum_{j=m}^{n-2} P(T_m, T_{j+1})\left(\beta_{j+1}^x - 1\right) - P(T_m, T_n) - K\sum_{j=m}^{n-1}\tau_j P(T_m, T_{j+1})\right)^+,$$

which is the payoff of a put option with strike $\beta_m^x$ on a bond

$$P^c(t) := \sum_{j=m}^{n-1} Y(T_j)P(t, T_{j+1}),$$

paying coupons $Y(T_j) := K\tau_j - (\beta_{j+1}^x - 1)$ at $T_{j+1}$, $j = m, \ldots, n-2$, and $Y(T_{n-1}) := K\tau_{n-1} + 1$ at $T_n$. For a corresponding zero-coupon bond to match the level of volatility, its remaining duration $D(t)$ must satisfy

$$B_x(D(t))^2 = \left(\sum_{j=m}^{n-1} w_j B_x(T_{j+1} - t)\right)^2,$$

with $B_x(\tau)$ as in Lemma 1 and where the $w_j := Y(T_j)P(t, T_{j+1})/(\sum_{k=m}^{n-1} Y(T_k)P(t, T_{k+1}))$ are the relative weights of the cashflows of the bond $P^c(t)$. The solution $D(t)$ has to be found numerically.

Also as in Trolle and Schwartz (2009) and Munk (1999) the swaption value can then be approximated by

$$V_{\text{pay}}(t) = \zeta P\left(t, T_m, t + D(t), \beta_m^x \zeta^{-1}\right),$$

with $\zeta = \frac{P^c(t)}{P(t, t+D(t))}$.

By put-call parity (Wu, 2019, ch. 6.4), the corresponding receiver swaption with equal payment schedule and fixed rate $K$ has the value

$$V_{\text{rec}}(t) = V_{\text{pay}}(t) - (S_t^x(T_m, T_n) - K)A(t).$$

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