South African Statistical Journal
Vol. 55, No. 2, pp. 87–108
https://doi.org/10.37920/sasj.2021.55.2.1

TIME-VARIANT NONPARAMETRIC EXTREME QUANTILE ESTIMATION WITH APPLICATION TO US TEMPERATURE DATA

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Statistical modelling for several years of daily temperature data is somewhat challenging due to remarkable variations of negative and positive temperatures throughout the year. A scatter plot of day and daily temperature shows the high magnitude of variations among data points as dots fall only in the first and fourth quadrants. One parametric modelling approach to this data is to use quantile regression to obtain regression lines on different quantiles. However, these quantile lines cannot make reliable predictions on extreme quantiles when time-variant quantiles differ significantly. In this paper, we develop several two-step nonparametric smoothing estimators and show their superiority over quantile regression for smoothing estimation of nonparametric quantiles with a novel application to temperature data. Narrower bootstrap confidence bands, smaller Minimum Absolute Distance (MAD), smaller bias and MSE, and higher coverage from the application and simulation results show that smoothing curves obtained from these smoothing estimators outperform the quantile regression line.

Keywords: Bandwidth, Kernel, Local Polynomials, Quantile Regression, Spline.

1. Introduction

Kernel smoothing, local polynomial smoothing and spline smoothing are very popular techniques for smoothing a smaller to moderate sized data set under the settings of nonparametric regression. However, these smoothing techniques become meaningless and yield very unstable results when (i) the size of the data is very large, (ii) the data itself shows spiky time-variant behaviour, and

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MSC2020 subject classifications. 62G08.

(iii) one-step smoothing is used to accommodate (i) and (ii). Data having attributes (i) and (ii) are usually smoothed on the first difference or second difference of the response variable to avoid over-smoothing and under-smoothing of the raw data, and statistical interpretations and prediction are made on the original scale by back transformation. On the other hand, when the purpose is to estimate and predict time-variant quantiles (e.g., minimum and maximum temperature of yearly data or other extreme quantiles), a small data set might not have enough time-variant quantiles with substantial variations among them. In this paper, we propose and develop three two-step nonparametric smoothing estimators for smoothing estimation of the time-variant extreme quantiles (empirical or nonparametric quantiles) with attributes (i) and (ii). Two-step smoothing estimators can easily accommodate big data in its estimation procedures if the purpose is to estimate time-variant unknown constants. More specifically, if the purpose is to estimate time variant smoothing quantiles or parameters from multiple years of data, then one-step smoothing (kernel log likelihood smoothing method) is not possible if the parametric structure of the response variable is not specified. A Kernel log likelihood smoothing estimate is obtained from the likelihood function multiplied by the kernel weighting function. Another one-step method, quantile regression, will not be able to make good approximations of the extreme quantiles when time-variant quantiles vary significantly. To overcome these estimation problems, two-step estimation procedure has been incorporated in the estimation procedure. Two-step estimation procedure consists of obtaining raw estimates of unknown quantiles by an empirical approach from the original data in the first step. These raw estimates of the time-variant quantiles are treated as the data points of the response variable and in the second step, the raw estimates are smoothed by applying smoothing estimators such as local polynomial, kernel or spline as derived in Section 3.

Nadaraya–Watson (Nadaraya, 1964; Watson, 1964) first developed and used kernel smoothing estimation, and since then it has been used in many applications, such as kernel density estimation (Silverman, 1986; Scott, 1992), kernel smoothing estimation of unknown functions (Hart and Wehrly, 1986), kernel smoothing estimation of distribution functions (Chowdhury et al., 2017, 2018), kernel smoothing estimation of time-variant parametric models (Chowdhury, 2017), two-step estimation of time-variant parameters and quantiles (Chowdhury et al., 2017), and estimation of time-varying coefficient models by a kernel estimator (Hoover et al., 1998). Local polynomial smoothing was first studied by Stone (1977, 1980, 1982) and Cleveland (1979) and then by Fan (1992, 1993), Fan and Gijbels (1992, 1996) and Ruppert and Wand (1994), among others. Smoothing splines have been studied by many authors, such as Schoenberg (1964), Reinsch (1967), Wahba (1975), and Silverman (1985), to name a few. See Eubank (1999) for a good review of spline methods and Wahba (1990) for a complete theoretical treatment.

In order to obtain smoothing nonparametric quantiles on the entire time range, we have derived three two-step smoothing estimators by modelling the time-variant raw estimates of the nonparametric quantiles. These estimators could be used when the parametric form of the response variable is unknown, the size of the data is big, or the data have significant variation by time points. For the estimation method, we first obtain time-variant raw estimates of the extreme quantiles at a set of distinct time points, and then compute the final estimators at any time point by smoothing the available raw estimates using these nonparametric smoothing estimators. We compare the relative performance of these smoothing estimators among themselves and with the quantile regression line by computing the MAD values of the observed and smoothed quantiles for the temperature data from seven US cities. We also construct their corresponding bootstrap confidence bands. All statistical computations and simulations are performed in R.

We describe time-varying nonparametric quantiles and parametric quantile regression in Section 2, and present our derivation of two-step smoothing estimators in Section 3. In Section 4, results from simulation studies are shown and an application of our procedures is presented in Section 5. Finally, we briefly discuss in Section 6 some further implications and extensions of the these procedures.

2. Quantiles and parametric quantile regression

Let $F_{Y_{t_j i}}(\cdot)$, $j = 1, 2, \ldots, J, i = 1, 2, \ldots, n_j$, be a time-variant distribution function. $Y_{t_j i}$ stands for ith observation of the t_jth year. More specifically, if $i = 79$ and $j = 11$, then $Y_{t_j i}$ stands for the 79th observation of the 11th year (there are up to 366 observations for each year). A value for the extreme quantile η for each t_j is estimated as

$$\tilde{\xi}_\eta(t_j) = \inf\{y_{t_j i} : F(y_{t_j i}) \geq \eta\} = F^{-1}(\eta), \tag{1}$$

where infimum is running over i and $\eta \in (0, 1)$. When $F(\cdot)$ does not belong to any parametric family, we can use the empirical version of $F(\cdot)$ to compute $\xi_\eta(t_j)$. We consider $\eta = 0.95$ and $\eta = 0.05$ in our application to US temperature data. The quantile regression estimator for quantile η minimises the objective function

$$Q(\beta_q) = \sum_{i:y_i \geq x_i'\beta} \eta|y_i - x_i'\beta_q| + \sum_{i:y_i \geq x_i'\beta} (1 - \eta)|y_i - x_i'\beta_q|. \tag{2}$$

This nondifferentiable function is minimised via the simplex method, which is guaranteed to yield a solution in a finite number of iterations. Interested readers can refer to the well-known paper of Koenker and Bassett (1978) for more on quantile regression.

3. Three nonparametric regressions

3.1 Estimation method

Our estimation is based on a two-step procedure in which we first split the sample space by a variable such as time or age, which is regarded as the explanatory variable in the nonparametric regression setting. For each split data, unknown statistical constants of interest, known as parameters, are estimated (method of moments, MLE, Bayesian methods or empirical methods) for each time point by engaging the response variable. These point-wise unrefined estimates are sometimes regarded as the raw estimates. In the second step, these unrefined estimates are smoothed by nonparametric regressions to obtain a predicted or smoothed value at any point on the entire time range. More specifically, suppose $\mathcal{S}$ is our sample space, which we split in J disjoint sets $\mathcal{S}_j$ by time variable t_j such that $\sum_{j=1}^{J} \mathcal{S}_j = \mathcal{S}$. By using the subjects in $\mathcal{S}_j$ at time $t_j \in \mathbf{t}$, we first estimate point-wise quantiles $\tilde{\xi}_\eta(t_j)$ of $\xi_\eta(t_j)$, and then derive the smoothing estimates of $\xi_\eta(t)$ for any $t \in \tau$ by applying the nonparametric regression over the corresponding $\tilde{\xi}_\eta(t_j)$ at each t_j. This two-step smoothing approach is computationally simple and can be used for both longitudinal data and time-variant cross-sectional data. For cross-sectional data, this procedure does not need correlation assumptions across different time points and for longitudinal data the correlation would be negligible if the repeated measurements appear in a manner of random long distant time points. By following this estimation method, we will derive the following three two-step nonparametric smoothing methods.

3.2 Two-step local polynomial smoothing regression

Suppose that $\xi_\eta(t)$ is $(p+1)$ times continuously differentiable with respect to $t \in \tau$. Let $\xi_\eta^{(q)}(t)$ be the qth derivative of $\xi_\eta(t)$, $1 \le q \le p$, and $\delta_q(t) = \xi_\eta^{(q)}(t)/q!$. By the Taylor expansion of $\xi_\eta(t)$, we have $\xi_\eta(t) \approx \sum_{q=0}^{p} \delta_q(a_0)(t_j - a_0)^q$ for t in some neighbourhood of a_0. We treat the raw estimates $\widetilde{\xi}_\eta(t_j)$ as the "observations" of $\xi_\eta(t_j)$ at t_j, and obtain the pth local polynomial estimators by minimising $\sum_{j=1}^{J}\{\widetilde{\xi}_\eta(t_j) - \sum_{q=0}^{p} \delta_q(t)(t_j - a_0)^q\}^2 K_h(t_j - a_0)$, where $K_h(t_j - a_0) = K[(t_j - a_0)/h]/h$, $K(\cdot)$ is a nonnegative kernel function, and $h > 0$ is a bandwidth. Using the matrix formulation, we define $\widetilde{\xi}_\eta(\mathbf{t}) = (\widetilde{\xi}_\eta(t_1), \ldots, \widetilde{\xi}_\eta(t_J))^T$, $\delta(t) = (\delta_0(t), \ldots, \delta_p(t))^T$, $G(t;h) = \text{diag}\{K_h(t_j - a_0)\}$ with jth column $G_j(t;h) = (0, \ldots, K_h(t_j - a_0), \ldots, 0)^T$, and $T_p(t)$ the $J \times (p+1)$ matrix with its jth row given by $T_{j,p}(t) = (1, t_j - a_0, \ldots, (t_j - a_0)^p)$. The local polynomial estimators $\widehat{\delta}_q(t)$ minimise $Q_G[\delta(t)] = [\widetilde{\xi}_\eta(\mathbf{t}) - T_p(t)\delta(t)]^T G(t;h)[\widetilde{\xi}_\eta(\mathbf{t}) - T_p(t)\delta(t)]$. The pth order local polynomial estimator of $\xi_\eta^{(q)}(t)$ based on $\widetilde{\xi}_\eta(t_j)$, which minimises $Q_G[\delta(t)]$, is

$$\widehat{\xi}_\eta^{(q)}(t) = \sum_{j=1}^{J} \Big\{ W_{q,p+1}(t_j, t; h)\, \widetilde{\xi}_\eta(t_j) \Big\}, \tag{3}$$

where $W_{q,p+1}(t_j, t; h) = q!e_{q+1,p+1}[T_p^T(t)G(t;h)T_p(t)]^{-1}[T_{j,p}^T(t)G_j(t;h)]$ is the "equivalent kernel function" (e.g., Fan and Zhang, 2000) and $e_{q+1,p+1}$ is the row vector of length $p+1$ with 1 as its $(q+1)$th entry and 0 elsewhere. By definition of $\delta(t)$, we have $\widehat{\delta}(t) = \big(\widehat{\delta}_0(t), \ldots, \widehat{\delta}_p(t)\big)^T$ and $\widehat{\xi}_\eta^{(q)}(t) = \widehat{\delta}_q(t)\, q!$ is an estimator of $\xi_\eta^{(q)}(t)$, $q = 0, 1, \ldots, p$. For local polynomial fitting $p - q$ should be taken to be odd as shown in Ruppert and Wand (1994) and Fan and Gijbels (1996). When $p = 1$, we get the local linear estimator $\widehat{\xi}_{\eta_L}(t) = \widehat{\delta}_0(t)$ of $\xi_\eta(t)$ based on (3) and the equivalent kernel function $W_{0,2}(t_j, t; h)$. So, the local linear estimator is

$$\widehat{\xi}_{\eta_L}(t) = \widehat{\xi}_\eta^{(0)}(t|x). \tag{4}$$

3.3 Two-step kernel smoothing regression

Suppose the random bivariate observations $(t_1, \xi_\eta(t_1)), \ldots, (t_J, \xi_\eta(t_J))$ each has joint density $f(t, \xi_\eta(t))$. Let $m(t)$ be an unknown function, which could be expressed by the nonparametric regression model:

$$\xi_\eta(t_j) = m(t_j) + \epsilon_j, \quad j = 1, \ldots, J, \tag{5}$$

where ϵ_j satisfies $E(\epsilon_j) = 0, Var(\epsilon_j) = \sigma_\epsilon^2$ and $Cov(\epsilon_j, \epsilon_k) = 0$ for $j \ne k$. Thus, we have

$$m(t) = E[\xi_\eta(t)|T = t] = \int \xi_\eta(t) f\big[\xi_\eta(t)|t\big] d\xi_\eta(t) = \frac{\int \xi_\eta(t) f\big[t, \xi_\eta(t)\big] d\xi_\eta(t)}{\int f\big[t, \xi_\eta(t)\big] d\xi_\eta(t)} = \frac{N}{D}. \tag{6}$$

$m(t)$ is a ratio of two correlated random terms. A product kernel density estimator technique will be used to estimate N and D separately. Therefore, by using the symmetry of the kernel and transformation of variables, we have

$$\hat{f}\big[t, \xi_\eta(t)\big] = \frac{1}{Jh_t h_{\xi_\eta}} \sum_{j=1}^{J} K\left(\frac{t - t_j}{h_t}\right) K\left(\frac{\xi_\eta(t) - \xi_\eta(t_j)}{h_{\xi_\eta}}\right) = \frac{1}{J}\sum_{j=1}^{J} K_{h_t}(t - t_j) K_{h_{\xi_\eta}}(\xi_\eta(t) - \xi_\eta(t_j)),$$

$$D = \int \hat{f}\big[t, \xi_\eta(t)\big] d\xi_\eta(t) = \frac{1}{J}\sum_{j=1}^{J} K_{h_t}(t - t_j) \int K_{h_{\xi_\eta}}(\xi_\eta(t) - \xi_\eta(t_j)) d\xi_\eta(t) = \frac{1}{J}\sum_{j=1}^{J} K_{h_t}(t - t_j) = \hat{f}(t),$$

$$N = \int \xi_\eta(t)\hat{f}\big[t, \xi_\eta(t)\big]d\xi_\eta = \frac{1}{J}\int \xi_\eta(t)\sum_{j=1}^{J} K_{h_t}(t-t_j)K_{h_{\xi_\eta}}(\xi_\eta(t)-\xi_\eta(t_j)) = \frac{1}{J}\sum_{j=1}^{J} K_{h_t}(t-t_j)\xi_\eta(t_j).$$

Therefore, we have

$$\hat{m}(t) = \sum_{j=1}^{J} W_{h_t}(t-t_j)\xi_\eta(t_j), \tag{7}$$

where $K_{h_t}(.) = K(.)/h_t$, $W_{h_t}(t-t_j) = K_{h_t}(t-t_j)/\sum_{j=1}^{J} K_{h_t}(t-t_j)$, and $\sum_{j=1}^{J} W_{h_t}(t-t_j) = 1$. Estimator (7) is widely known as the Nadaraya–Watson kernel estimator. h is known as the bandwidth or smoothing parameter.

3.4 Two-step spline smoothing regression

Let us consider the data points $(t_1, \xi_\eta(t_1)), (t_2, \xi_\eta(t_2)), \ldots, (t_J, \xi_\eta(t_J))$. We want to find a function $\hat{m}(t)$, which is a good approximation to the true regression function $m(t) = E(\xi_\eta(t)|T = t)$. A natural way to do this is to minimise the spline objective function

$$O(m,h) = \sum_{j=1}^{J}\big(\xi_\eta(t_j) - m(t_j)\big)^2 + h\int \big(m''(t)\big)^2 dt, \tag{8}$$

where h is a smoothing parameter, chosen by cross-validation. The first term is just the mean squared error (MSE) using the curve $m(t)$ to predict $\xi_\eta(t)$. The second term penalises curvature in the function. m'' is the second derivative of m with respect to t, which confirms the existence of curvature of m at t. The sign of m'' tells whether m is concave or convex but squaring it makes it immaterial. Integration of this over all t determines the average curvature of m. Finally, this is multiplied by h and added to the MSE. Given two functions with the same MSE, we choose the one with less average curvature. It can be shown (Green and Silverman, 1994; Solo, 1999) that (8) has an explicit, finite-dimensional, unique minimiser which is a natural cubic spline with knots at the unique values of the t_j, $j = 1, 2, \ldots, J$. It seems that the family is still over-parametrised, since there are as many as J knots, which implies J degrees of freedom. However, the penalty term translates to a penalty on the spline coefficients, which are shrunk some of the way toward a linear fit (Hastie et al., 2009). Since the solution is a natural spline, we can write it as $m(t) = \sum_{j=1}^{J} N_j(t)\theta_j$, where the N_j are a J-dimensional set of basis functions for representing this family of natural splines. After the above reparametrisation, the optimisation problem (8) turns out to be

$$O(\theta,h) = \sum_{j=1}^{J}\left(\xi_\eta(t_j) - \sum_{j=1}^{J} N_j(t)\theta_j\right)^2 + h\int\left(\sum_{j=1}^{J} N_j''(t)\theta_j\right)^2 dt. \tag{9}$$

By defining the basis matrix and penalty matrices N and $\Omega \in \Re$ by $N_{ij} = N_j(t_i)$ and $\Omega_{ij} = \int N_i''(t)N_j''(t)dt$, for $i, j = 1, 2, \ldots, J$, problem (9) becomes

$$O(\theta,h) = (\xi_\eta - N\theta)^T(\xi_\eta - N\theta) + h\theta^T\Omega\theta. \tag{10}$$

The solution is easily seen to be $\tilde{\theta} = (N^TN + h\Omega)^{-1}N^T\xi_\eta$, with fitted smoothing spline

$$\hat{m}(t) = N(N^TN + h\Omega)^{-1}N^T\xi_\eta = \sum_{j=1}^{J} N_j(t)\tilde{\theta}_j. \tag{11}$$

3.5 Minimum absolute distance (MAD) values

Suppose $\xi_\eta(t_j)$ are the observed values of the nonparametric quantile of order η at time point t_j. In our application, we consider $\eta = 0.95$ and $\eta = 0.05$, which stand for 95th and 5th percentile values. Let $\hat{\xi}_{\eta LP}(t_j)$, $\hat{\xi}_{\eta KS}(t_j)$, $\hat{\xi}_{\eta SS}(t_j)$ and $\hat{\xi}_{\eta QR}(t_j)$ be the smoothed/fitted values obtained from local polynomial smoothing, kernel smoothing, spline smoothing and quantile regression. The MAD values for each of the three smoothing estimates with respect to the quantile regression estimate for each time point are computed by $|\hat{\xi}_{\eta LP}(t_j) - \xi_\eta(t_j)|$, $|\hat{\xi}_{\eta KS}(t_j) - \xi_\eta(t_j)|$, $|\hat{\xi}_{\eta SS}(t_j) - \xi_\eta(t_j)|$ and $|\hat{\xi}_{\eta QR}(t_j) - \xi_\eta(t_j)|$ for $j = 1, \ldots, J$. In Section 5, we compare the MAD values for each of the seven cities and select the Best Estimator (BE). The BE is chosen as the estimator with the smallest MAD value. The BE refers to the estimator that approximates the observed quantiles best.

3.6 Kernel selection

In nonparametric regression such as local polynomial smoothing and kernel smoothing, the kernel works as a weighting function. Similar to density estimation, kernel regression uses a kernel function $K : \Re \to \Re$, satisfying $\int K(x)dx = 1$, $\int xK(x)dx = 0$, $0 \le \int x^2 K(x)dx \le \infty$. The Gaussian kernel and the Epanechnikov kernel are two commonly used kernels respectively defined by $K(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2)$, $x \in \Re$, and $K(x) = \frac{3}{4}(1 - x^2)$, $|x| \le 1$. MISE (Mean Integrated Squared Error) or AMISE (Asymptotic MISE) are two metrics to check the comparative performance of the kernels. The Epanechnikov kernel minimises AMISE, and is therefore considered optimal. The Epanechnikov kernel is used in our application and simulation studies. In nonparametric regression, kernel selection is not as important as the choice of bandwidth. No kernel is used in the two-step spline smoothing estimator.

3.7 Cross-validation for bandwidth choices

In nonparametric regression, the bandwidth controls the smoothness and roughness of the smoothing estimator. Two popular bandwidth selection techniques are "Leave-One-Subject-Out Cross Validation (LSCV)" and "Leave-One-Time-Point-Out Cross Validation (LTCV)." The LSCV procedure deletes observations one at a time while LTCV deletes all observations at the time design points $\mathbf{t} = (t_1, \ldots, t_J)$. The bandwidths for (4) and (7) and smoothing parameters for (11) are selected using the LTCV procedure because our data in the applications and simulations are binned to different time (age) points. The cross-validation criterion is

$$CV(h) = \sum_{j=1}^{J} \sum_{i \in \mathcal{S}_j} W_i \left\{ Y_i(t_j) - \widehat{\xi}_\eta^{(-j)}(t_j) \right\}^2, \tag{12}$$

where W_i is a weight function which could be $1/N$ and $\widehat{\xi}_\eta^{(-j)}(t_j)$ is the nonparametric regression estimators of (3), (7) and (11) applied to the data at all time points except time point t_j. The CV choice of h is the one that minimises $CV(h)$ over $h \ge 0$. Bandwidth choice plays a significant role in nonparametric regression. A subjective or wrong choice of very small ($h \to 0$) or very large ($h \to \infty$) bandwidth will produce undersmoothed or oversmoothed curves. For a very large choice of bandwidth, nonparametric smoothing estimates converge to the ordinary least squares fit of a straight line yielding higher biases in smoothing curve. On the other hand, if the bandwidth is very small, the smoothing estimates will have large variances.

3.8 Point-wise bootstrap confidence band

Hoover et al. (1998) suggested the "resampling-subject" bootstrap for inferences on nonparametric analysis. By incorporating his suggestion to our two-step estimation procedure, we can obtain a point-wise bootstrap confidence band for $\xi_\eta(t)$ by first obtaining B bootstrap samples through resampling the subjects from the sample with replacement, and then computing B two-step smoothing estimators $\{\widehat{\xi}^b_\eta(t) : b = 1, \dots B\}$ using (4), (7) and (11) for each of the bootstrap samples. A similar procedure is followed to construct bootstrap confidence bands for the quantile regression line. The lower and upper boundaries of the $[100 \times (1-\alpha)]$th empirical quantile bootstrap point-wise confidence band of $\widehat{\xi}_\eta(t)$ are the empirical lower and upper $[100 \times (\alpha/2)]$th percentiles based on the bootstrap estimators $\{\widehat{\xi}^b_\eta(t) : b = 1, \dots B\}$. Alternatively, if $SD\{\widehat{\xi}^b_\eta(t)\}$ is the empirical standard deviation of $\left\{\widehat{\xi}^b_\eta(t) : b = 1, \dots B\right\}$, the $[100 \times (1-\alpha)]$th normally approximated bootstrap pointwise confidence interval of $\widehat{\xi}_\eta(t)$ is $\widehat{\xi}_\eta(t) \pm Z_{1-\alpha/2} \times SD\left\{\widehat{\xi}^b_\eta(t)\right\}$, where $Z_{1-\alpha/2}$ is the $[100 \times (1-\alpha/2)]$th percentile value coming from the standard normal distribution.

4. Simulation

In this section, we conduct a simulation study to assess the performances of the smoothing curves obtained from the two-step smoothing estimators against the quantile regression line. We also compare the relative performance of these three two-step smoothing estimators among themselves. We compare their performances by computing BIAS, MSE, and COVERAGE. Data are simulated with increasing variance over 50 time points (TP) with the standard deviation s_t being $s_t = 0.1 + 0.05t$, $t \in \{1, 2, ..., 50\}$. The heterocedastic model for data simulation is $y = b_0 + b_1 t + e$, where $b_0 = 3$, $b_1 = 0.1$ and $e \sim N(0, s_t)$. We generated 500 simulated data to evaluate these four methods on the curve/line estimation for 95th and 5th percentile values. The Epanechnikov kernel and the optimal bandwidth from cross-validation are used for the smoothing estimators. We then calculate MSE, BIAS and COVERAGE to determine which method is best. Table 1 and Table 2 represent the simulation results for the 95th and 5th percentile values respectively. From Table 1, we see that the two-step local polynomial smoothing (LP) and the two-step spline smoothing (SS) have less bias than the quantile regression (QR) line for all 50 time points. For the two-step kernel smoothing (KS), we see that only at first four time points, QR has less bias than the KS. For comparison of MSE for these four methods, we conclude that the SS estimator has less MSE than QR at all 50 time points. We also see that at the first three time points, the KS estimator has a higher MSE than QR and only in the last time point, the LP smoothing estimator has a higher MSE than QR. In all other time points, QR has a higher MSE than then the LP estimator and the KS estimator. For coverage probability, we see that only at time point 49, QR has a higher coverage than the LP estimator and in all other time points, QR has a lower coverage than all three two-step smoothing estimators. From Table 1, we also see that out of the 50 time points, LP has less bias in 17 time points whereas KS and SS have less bias in 18 and 15 time points respectively. In terms of MSE, we see from Table 1 that the KS has a less MSE in 41 time points than the SS and the LP. In terms of Coverage probability, all three smoothing estimators have consistent results. Similar explanations stand for Table 2.

Table 1. Bias, MSE, Coverage and Best Estimator (BE) corresponding to 50 Time Points (TP) for the Quantile Regression (QR), Local Polynomial Smoothing (LP), Kernel Smoothing (KS) and Spline Smoothing (SS) estimators for the 95th percentile values from the simulation design.

	Bias					MSE					Coverage				
TP	**QR**	**LP**	**KS**	**SS**	**BE**	**QR**	**LP**	**KS**	**SS**	**BE**	**QR**	**LP**	**KS**	**SS**	**BE**
1	0.064 25	0.004 63	0.399 15	0.006 69	lp	0.014 23	0.001 66	0.183 66	0.009 41	lp	0.920	0.945	0.245	0.960	ss
2	0.061 18	−0.005 38	0.262 46	−0.003 19	ss	0.016 54	0.003 71	0.088 19	0.010 92	lp	0.930	0.995	0.500	0.990	lp
3	0.071 29	−0.007 91	0.165 25	−0.005 43	ss	0.020 93	0.007 52	0.044 64	0.013 55	lp	0.935	0.995	0.730	0.980	lp
4	0.089 26	0.007 39	0.110 77	0.010 33	lp	0.028 57	0.010 01	0.026 27	0.012 72	lp	0.895	0.985	0.845	0.975	lp
5	0.100 44	0.012 34	0.071 10	0.014 93	lp	0.035 54	0.015 30	0.021 87	0.019 81	lp	0.930	0.985	0.910	0.975	lp
6	0.081 92	−0.000 08	0.031 12	0.000 78	lp	0.037 98	0.020 53	0.021 49	0.023 52	lp	0.960	0.995	0.995	0.975	lp
7	0.079 86	−0.007 25	0.009 92	−0.008 23	lp	0.041 77	0.022 15	0.021 71	0.034 07	ks	0.950	0.995	0.980	0.995	lp, ss
8	0.096 06	0.004 26	0.014 23	0.001 54	ss	0.056 11	0.029 46	0.028 56	0.036 57	ks	0.960	0.995	0.990	0.990	lp
9	0.102 21	−0.012 41	−0.008 84	−0.016 01	ks	0.054 71	0.030 35	0.028 98	0.052 34	ks	0.950	0.980	0.975	0.985	ss
10	0.115 24	−0.002 87	−0.003 89	−0.006 82	lp	0.065 38	0.033 27	0.031 73	0.056 33	ks	0.920	0.990	0.990	0.990	lp, ks, ss
11	0.133 42	0.013 99	0.013 29	0.009 81	ss	0.087 80	0.046 18	0.044 45	0.053 24	ks	0.930	0.985	0.980	0.990	ss
12	0.117 55	−0.006 24	−0.004 98	−0.010 68	ks	0.094 08	0.053 36	0.051 44	0.061 82	ks	0.970	0.980	0.980	0.990	ss
13	0.127 93	−0.014 82	−0.017 08	−0.018 74	lp	0.101 73	0.058 83	0.057 39	0.082 69	ks	0.940	0.990	0.985	0.995	ss
14	0.152 21	0.003 83	0.007 03	0.001 22	ss	0.113 69	0.068 22	0.066 13	0.096 00	ks	0.940	1.000	0.995	1.000	lp, ss
15	0.175 67	0.023 65	0.019 60	0.023 77	ks	0.144 88	0.082 65	0.079 63	0.121 56	ks	0.950	0.990	0.990	0.985	lp, ks
16	0.168 39	0.002 92	0.002 74	0.005 84	ks	0.147 88	0.092 33	0.087 65	0.115 06	ks	0.960	0.990	0.990	0.995	ss
17	0.126 75	−0.038 80	−0.039 80	−0.035 25	ss	0.125 54	0.088 92	0.086 53	0.133 31	ks	0.960	0.995	0.995	0.995	lp, ks, ss
18	0.192 63	0.020 17	0.020 42	0.025 12	lp	0.178 32	0.101 30	0.095 18	0.136 19	ks	0.930	0.985	0.980	0.985	lp, ss
19	0.184 31	−0.004 35	−0.003 79	0.002 74	ss	0.208 63	0.127 56	0.122 67	0.142 66	ks	0.965	0.985	0.985	0.995	ss
20	0.191 42	0.012 31	0.011 66	0.017 88	ks	0.211 68	0.130 18	0.124 31	0.148 24	ks	0.955	0.990	0.985	0.995	ss
21	0.185 47	−0.005 95	−0.004 89	−0.004 95	ks	0.301 45	0.189 34	0.181 63	0.176 52	ss	0.975	0.995	0.995	0.995	lp, ks, ss
22	0.196 90	0.001 05	0.003 62	−0.000 04	ss	0.236 81	0.160 01	0.154 40	0.228 22	ks	0.965	0.995	0.995	0.995	lp, ks, ss
23	0.197 82	0.000 85	−0.003 42	0.000 94	lp	0.272 54	0.170 30	0.164 08	0.198 83	ks	0.960	0.990	0.985	0.995	ss
24	0.226 26	0.002 78	0.007 69	0.004 35	lp	0.288 84	0.178 87	0.169 10	0.274 31	ks	0.965	0.985	0.985	1.000	ss
25	0.222 51	0.001 81	0.000 49	−0.000 34	ss	0.279 59	0.168 15	0.163 04	0.231 27	ks	0.960	0.995	0.995	0.995	lp, ks, ss
26	0.232 25	−0.002 49	−0.000 77	−0.011 38	ks	0.312 66	0.200 60	0.189 93	0.270 53	ks	0.940	0.990	0.990	0.995	ss
27	0.227 57	−0.011 18	−0.014 84	−0.020 07	lp	0.388 42	0.245 13	0.236 73	0.300 00	ks	0.970	0.995	0.995	1.000	ss
28	0.270 96	0.020 97	0.015 47	0.019 48	ks	0.409 51	0.237 71	0.232 56	0.359 96	ks	0.930	0.990	0.990	0.995	ss
29	0.259 05	0.006 01	0.009 72	0.010 03	lp	0.412 58	0.260 56	0.252 14	0.354 65	ks	0.940	0.995	0.995	0.985	lp, ks
30	0.246 40	−0.014 57	−0.004 28	−0.011 01	ks	0.459 18	0.298 59	0.278 81	0.431 37	ks	0.960	0.985	0.985	0.985	lp, ks, ss
31	0.255 85	−0.004 39	−0.003 22	−0.004 16	ks	0.537 05	0.341 62	0.323 36	0.395 22	ks	0.975	0.995	0.995	1.000	ss
32	0.311 17	0.048 42	0.042 69	0.043 96	ks	0.458 38	0.271 80	0.264 32	0.378 54	ks	0.940	0.975	0.975	0.985	ss
33	0.171 32	−0.088 65	−0.093 48	−0.099 57	lp	0.558 13	0.377 77	0.365 95	0.427 12	ks	0.985	0.995	0.995	1.000	ss
34	0.328 32	0.076 17	0.075 05	0.060 25	ss	0.544 38	0.327 52	0.315 62	0.444 87	ks	0.945	0.990	0.985	0.990	lp, ss
35	0.211 72	−0.048 10	−0.044 56	−0.061 64	ks	0.567 33	0.379 44	0.369 00	0.520 47	ks	0.970	0.985	0.985	0.990	ss
36	0.265 38	−0.012 43	−0.008 36	−0.016 07	ks	0.524 41	0.355 80	0.346 31	0.459 28	ks	0.965	0.985	0.985	0.985	lp, ks, ss
37	0.336 27	0.038 86	0.038 86	0.044 57	ks	0.660 87	0.375 24	0.358 00	0.491 19	ks	0.980	0.990	0.990	0.985	lp, ks
38	0.324 62	0.008 68	0.008 11	0.016 96	ks	0.723 91	0.470 46	0.453 72	0.628 37	ks	0.985	0.985	0.985	0.995	ss
39	0.284 20	−0.031 08	−0.037 20	−0.022 79	ss	0.746 61	0.472 26	0.441 82	0.646 82	ks	0.975	0.990	1.000	0.990	ks
40	0.374 69	0.046 80	0.042 11	0.059 66	ks	0.780 38	0.489 31	0.468 29	0.664 90	ks	0.935	0.990	0.990	0.995	ss
41	0.334 26	0.008 81	0.004 29	0.026 40	ks	0.760 65	0.472 13	0.454 59	0.639 80	ks	0.955	0.985	0.985	0.980	lp, ks
42	0.306 12	−0.028 24	−0.023 77	−0.014 19	ss	0.767 04	0.548 03	0.514 63	0.657 94	ks	0.990	0.995	0.995	0.995	lp, ks, ss
43	0.330 58	0.004 18	−0.008 81	0.012 46	lp	0.799 22	0.546 40	0.531 84	0.784 88	ks	0.945	0.985	0.985	0.985	lp, ks, ss
44	0.237 98	−0.086 51	−0.092 65	−0.077 47	ss	0.871 77	0.623 92	0.591 79	0.706 85	ks	0.985	0.995	0.995	0.995	lp, ks, ss
45	0.384 79	0.056 59	0.027 10	0.064 49	ks	0.920 05	0.587 82	0.551 79	0.684 89	ks	0.940	0.985	0.990	0.985	ks
46	0.325 14	−0.012 38	−0.079 62	−0.016 48	lp	0.881 25	0.592 29	0.574 85	0.726 94	ks	0.965	0.975	0.985	0.985	ks, ss
47	0.319 18	−0.025 65	−0.126 81	−0.044 49	lp	0.815 39	0.571 40	0.551 09	0.652 86	ks	0.950	0.975	0.980	0.980	ks, ss
48	0.399 23	0.046 61	−0.133 83	0.020 65	ss	1.294 77	0.785 62	0.870 90	0.710 95	ss	0.980	0.980	0.990	0.995	ss
49	0.341 17	−0.016 70	−0.299 60	−0.044 20	lp	1.304 80	0.342 84	0.863 44	0.750 18	lp	0.980	0.975	1.000	1.000	ks, ss
50	0.438 77	0.082 83	−0.327 00	0.048 40	ss	0.988 80	2.069 27	0.623 01	0.738 52	ks	0.935	0.975	1.000	0.980	ks

Table 2. Bias, MSE, Coverage and Best Estimator (BE) corresponding to 50 Time Points (TP) for the Quantile Regression (QR), Local Polynomial Smoothing (LP), Kernel Smoothing (KS) and Spline Smoothing (SS) estimators for the 5th percentile values of the simulation design.

	Bias for Maximum Smoothing Values					MSE for Maximum Smoothing Values					Coverage for Maximum Smoothing Values				
TP	QR	LP	KS	SS	BE	QR	LP	KS	SS	BE	QR	LP	KS	SS	BE
1	−0.033 67	−0.001 54	−0.186 87	−0.000 02	ss	0.007 11	0.003 40	0.044 44	0.009 41	lp	0.935	0.935	0.480	0.945	ss
2	−0.031 56	0.007 54	−0.151 30	0.008 51	lp	0.009 86	0.005 03	0.033 92	0.010 92	lp	0.940	0.965	0.660	0.940	lp
3	−0.049 70	−0.003 11	−0.137 66	−0.002 14	ss	0.014 63	0.006 77	0.030 79	0.013 55	lp	0.910	0.945	0.715	0.945	lp, ss
4	−0.067 66	−0.010 07	−0.122 92	−0.009 58	ss	0.017 54	0.010 43	0.030 44	0.012 72	lp	0.905	0.950	0.810	0.955	ss
5	−0.071 63	−0.007 74	−0.100 91	−0.006 86	ss	0.023 47	0.014 41	0.028 19	0.019 81	lp	0.920	0.955	0.880	0.955	lp, ss
6	−0.076 29	−0.007 35	−0.085 31	−0.006 10	ss	0.029 27	0.018 88	0.028 51	0.023 52	lp	0.930	0.960	0.915	0.955	lp
7	−0.053 37	0.019 58	−0.045 56	0.022 41	lp	0.037 41	0.029 48	0.031 63	0.034 07	lp	0.955	0.940	0.960	0.950	ks
8	−0.088 11	−0.006 04	−0.058 39	−0.002 21	ss	0.044 19	0.032 79	0.036 62	0.036 57	lp	0.925	0.960	0.940	0.960	lp, ss
9	−0.067 36	0.020 24	−0.026 17	0.023 67	lp	0.055 82	0.048 95	0.047 94	0.052 34	ks	0.950	0.935	0.940	0.940	qr
10	−0.085 18	0.008 86	−0.024 79	0.012 44	lp	0.065 34	0.054 37	0.053 74	0.056 33	ks	0.945	0.950	0.960	0.960	ks, ss
11	−0.109 88	−0.013 22	−0.038 43	−0.006 12	ss	0.065 90	0.050 51	0.052 08	0.053 24	lp	0.930	0.955	0.960	0.960	ks, ss
12	−0.121 29	−0.016 06	−0.035 10	−0.008 74	ss	0.079 07	0.056 59	0.058 72	0.061 82	lp	0.930	0.960	0.965	0.955	ks
13	−0.102 67	0.011 14	0.000 41	0.018 93	ks	0.097 01	0.075 97	0.072 86	0.082 69	ks	0.940	0.940	0.945	0.950	ss
14	−0.131 73	−0.009 15	−0.019 40	−0.002 73	ss	0.117 68	0.090 62	0.087 52	0.096 00	ks	0.940	0.945	0.950	0.950	ks, ss
15	−0.078 83	0.053 30	0.039 09	0.058 14	ks	0.128 69	0.114 45	0.108 20	0.121 56	ks	0.950	0.930	0.935	0.940	qr
16	−0.160 68	−0.018 83	−0.022 57	−0.014 86	ss	0.145 51	0.107 46	0.106 06	0.115 06	ks	0.930	0.960	0.960	0.955	lp, ks
17	−0.185 85	−0.033 10	−0.035 28	−0.032 28	ss	0.172 10	0.127 72	0.126 75	0.133 31	ks	0.945	0.955	0.955	0.945	lp, ks
18	−0.126 49	0.034 92	0.036 25	0.034 53	ss	0.159 19	0.126 95	0.124 89	0.136 19	ks	0.950	0.955	0.955	0.950	lp, ks
19	−0.167 38	−0.001 04	−0.000 28	−0.003 61	ks	0.175 50	0.134 61	0.132 79	0.142 66	ks	0.935	0.965	0.965	0.965	lp, ks, ss
20	−0.213 36	−0.039 18	−0.040 67	−0.041 32	lp	0.194 60	0.142 96	0.141 67	0.148 24	ks	0.920	0.960	0.960	0.950	lp, ks
21	−0.208 26	−0.025 72	−0.025 58	−0.031 11	ks	0.227 48	0.166 17	0.160 42	0.176 52	ks	0.925	0.955	0.950	0.955	lp, ss
22	−0.190 34	0.000 09	−0.008 79	−0.006 55	lp	0.281 00	0.206 55	0.199 72	0.228 22	ks	0.930	0.960	0.960	0.960	lp, ks, ss
23	−0.172 07	0.027 16	0.025 03	0.019 44	ss	0.240 29	0.178 15	0.172 81	0.198 83	ks	0.940	0.955	0.955	0.950	lp, ks
24	−0.226 89	−0.018 76	−0.015 60	−0.028 87	ks	0.338 13	0.247 54	0.244 41	0.274 31	ks	0.940	0.960	0.960	0.960	lp, ks, ss
25	−0.218 40	−0.002 86	−0.004 50	−0.015 33	lp	0.290 43	0.208 56	0.204 04	0.231 27	ks	0.955	0.970	0.970	0.960	lp, ks
26	−0.188 01	0.035 84	0.040 05	0.025 43	ss	0.321 43	0.243 74	0.241 32	0.270 53	ks	0.955	0.965	0.965	0.960	lp, ks
27	−0.209 26	0.021 29	0.018 28	0.012 49	ss	0.375 58	0.284 66	0.277 94	0.300 00	ks	0.935	0.950	0.945	0.950	lp, ss
28	−0.286 59	−0.059 18	−0.060 19	−0.066 87	lp	0.465 18	0.325 70	0.318 87	0.359 96	ks	0.930	0.950	0.950	0.945	lp, ks
29	−0.238 12	−0.010 65	−0.005 10	−0.015 03	ks	0.437 35	0.339 31	0.333 47	0.354 65	ks	0.970	0.975	0.975	0.980	ss
30	−0.219 45	0.009 01	0.011 32	0.009 43	lp	0.525 76	0.388 30	0.381 33	0.431 37	ks	0.940	0.955	0.955	0.960	ss
31	−0.270 46	−0.035 59	−0.033 87	−0.036 70	ks	0.496 02	0.355 33	0.346 17	0.395 22	ks	0.950	0.945	0.940	0.955	ss
32	−0.275 83	−0.032 76	−0.038 47	−0.038 09	lp	0.504 73	0.356 21	0.340 76	0.378 54	ks	0.945	0.955	0.955	0.965	ss
33	−0.214 67	0.036 59	0.035 76	0.032 13	ss	0.487 65	0.404 85	0.401 75	0.427 12	ks	0.935	0.955	0.955	0.965	ss
34	−0.255 68	0.005 47	0.013 76	0.004 90	ss	0.524 70	0.412 91	0.407 70	0.444 87	ks	0.960	0.950	0.950	0.950	qr
35	−0.229 57	0.040 15	0.051 24	0.039 87	ss	0.615 95	0.487 54	0.478 33	0.520 47	ks	0.945	0.950	0.945	0.950	lp, ss
36	−0.298 29	−0.022 84	−0.026 47	−0.026 87	lp	0.561 04	0.420 90	0.407 15	0.459 28	ks	0.935	0.950	0.955	0.955	ks, ss
37	−0.297 26	−0.017 95	−0.008 89	−0.020 40	ks	0.596 51	0.466 15	0.461 67	0.491 19	ks	0.935	0.955	0.955	0.950	lp, ks
38	−0.201 12	0.080 33	0.099 16	0.079 13	ss	0.723 35	0.584 60	0.577 47	0.628 37	ks	0.955	0.945	0.940	0.960	ss
39	−0.288 14	−0.003 02	0.023 25	−0.008 39	lp	0.790 41	0.602 80	0.584 07	0.646 82	ks	0.940	0.945	0.940	0.950	ss
40	−0.216 61	0.077 73	0.088 03	0.066 60	ss	0.767 66	0.628 87	0.611 74	0.664 90	ks	0.955	0.940	0.920	0.930	qr
41	−0.240 84	0.072 48	0.097 05	0.063 46	ss	0.753 35	0.604 92	0.597 11	0.639 80	ks	0.950	0.955	0.955	0.950	lp, ks
42	−0.325 61	0.009 85	0.037 26	−0.003 12	ss	0.802 73	0.609 52	0.566 09	0.657 94	ks	0.950	0.960	0.960	0.965	ss
43	−0.388 27	−0.035 10	0.006 27	−0.054 14	ks	0.973 16	0.736 78	0.720 60	0.784 88	ks	0.955	0.950	0.940	0.965	ss
44	−0.390 09	−0.029 91	0.035 40	−0.043 09	lp	0.901 56	0.675 17	0.655 64	0.706 85	ks	0.950	0.960	0.965	0.960	ks
45	−0.342 02	0.015 89	0.086 24	0.012 97	ss	0.898 33	0.627 19	0.626 99	0.684 89	ks	0.950	0.945	0.945	0.950	qr, ss
46	−0.361 46	−0.009 44	0.064 40	−0.010 74	lp	0.967 00	0.693 63	0.673 28	0.726 94	ks	0.955	0.945	0.935	0.950	qr
47	−0.380 94	−0.027 25	0.067 02	−0.021 64	ss	0.913 49	0.542 09	0.584 94	0.652 86	lp	0.945	0.950	0.940	0.935	lp
48	−0.418 14	−0.053 62	0.061 09	−0.050 10	ss	0.971 18	0.583 49	0.597 32	0.710 95	lp	0.935	0.935	0.940	0.950	ss
49	−0.344 33	0.045 02	0.183 37	0.031 88	ss	0.961 01	0.517 63	0.761 24	0.750 18	lp	0.955	0.950	0.940	0.950	qr
50	−0.346 73	0.077 28	0.201 69	0.037 28	ss	0.996 87	1.357 17	0.765 60	0.738 52	ss	0.940	0.950	0.940	0.945	lp

5. Application to temperature data in seven US cities

We apply our methods to temperature data measured in degree Celsius from seven US cities. The cities were selected in such a way that three of them are in the extreme north (Minneapolis, Portland and Seattle), three are in the extreme south (Dallas, Miami, San Diego) and one (Kansas) is in the middle of the US. These data were recorded on each day by the US Meteorological Department from 1990 to 2016. From this data, we computed the 95th and 5th percentile temperature for each of the 27 years for these 7 cities. We have $J = 27$ distinct time design points $\{t_1, t_2, \ldots, t_{27}\} = \{1990, 1991, \ldots, 2016\}$. Thus, for a given $1 \leq j \leq J = 27$, we denote $T_{0.95}(t_j)$ and $T_{0.05}(t_j)$ as the 95^{th} and 5^{th} percentile values of temperature at year t_j. The values of $T_{0.95}(t_j)$ and $T_{0.05}(t_j)$ are regarded as the raw estimate for each t_j. Applying the two-step local polynomial smoothing (LP) estimator (4), kernel smoothing (KS) estimator (7), spline smoothing (SS) estimator (11) to the quantiles $\{T_{0.95}(t_j), t_j; 1 \leq j \leq J, 1 \leq i \leq n\}$ and $\{T_{0.05}(t_j), t_j; 1 \leq j \leq J, \}$, we obtain the 95th and 5th smoothing quantile curves on temperature data for any time point within the entire time design points $\{t_1, t_2, \ldots, t_{27}\} = \{1990, 1991, \ldots, 2016\}$. It should be noted that the fitted quantile regression (QR) line from equation (2) is obtained from the entire data $Y_{t_j i}$.

Figures 1 and 2 show the KS estimates, LP estimates, SS estimates, and QR estimates of $T_{0.95}(t_j)$ and $T_{0.05}(t_j)$ and their corresponding bootstrap pointwise 95% confidence bands based on B=500 bootstrap replications. The Epanechnikov kernel was used as a weighting function for KS and LP smoothing. In Figure 1, KS T0.95 DAL, LPS T0.95 DAL, SS T0.95 DAL and QR T0.95 DAL stand for two-step kernel smoothing estimates, two-step local polynomial smoothing estimates, two-step spline smoothing estimates, and quantile regression estimates of $T0.95(t_j)$ in Dallas. Similar abbreviations are used for other cities corresponding to $T_{0.95}(t_j)$ and $T_{0.05}(t_j)$ in Figures 1 and 2.

The value of the bandwidth h was chosen by minimising the LTCV scores. One concern when choosing the optimal h in this application is that a range of h was set in advance. This is because a too large h will flatten the smooth curve and fail to catch the “curvature" pattern in the original data while a too small h will make the smooth curve too spiky. Therefore, a range of 1 to 10 of h was used for KS and LP estimates to look for the value that can minimise the LTCV scores, while a range of 0 to 1.5 was used for SS estimates of parameter λ. Furthermore, to avoid getting unusual estimations near the boundary of the sample data (close to 1990 or 2016), some observations that were close to the boundary were removed when comparing the LTCV scores. A parameter named TRIMMED was used to control the number of observations removed. For instance, TRIMMED = 1 means that the first and last observations were removed when comparing the LTCV scores. Since there are only 27 observations for each city, the value of TRIMMED was controlled within 3 in this application. Tables 3 and 4 show the values of h and TRIMMED for KS estimates, LP estimates, and SS estimates of $T_{0.95}(t_j)$ and $T_{0.05}(t_j)$ for each of the 7 cities. In Table 3, KS_h and KS_Trim stand for values of bandwidth h and TRIMMED for two-step kernel smoothing estimates of $T_{0.95}(t_j)$. LP_h and LP_Trim stand for values of bandwidth h and TRIMMED for two-step local polynomial smoothing estimates of $T_{0.95}(t_j)$. SS_h and SS_Trim stand for values of bandwidth h and TRIMMED for spline smoothing estimates of $T_{0.95}(t_j)$. Similar abbreviations stand for the $T0.05(t_j)$ in Table 4.

Bootstrap confidence bands have been calculated to demonstrate that the bandwidth choice is made correctly and also to see which smoothing estimator has narrower confidence bands. In Figures 1 and 2, dots represent the raw estimates, solid black lines represent smoothing estimates and dashed lines represent the 95% pointwise bootstrap confidence bands of $T_{0.95}(t_j)$ and $T_{0.05}(t_j)$. By looking

Table 3. Values of the bandwidth h and TRIMMED for the 7 cities for the kernel smoothing estimate, local polynomial smoothing estimate, and spline smoothing estimate of $T0.95(t_j)$.

City	KS_h	KS_Trim	LP_h	LP_Trim	SS_h	SS_Trim
DAL	1.15	3	1.01	3	0.25	3
KANSAS	1.02	3	1.01	3	0.24	3
MIAMI	10.01	3	10.01	3	0.99	3
MINNEAPOLIS	1.18	3	1.01	3	0.20	3
PORTLAND	1.26	3	1.01	3	0.26	3
SAN DIEGO	2.80	3	1.04	3	0.68	3
SEATTLE	3.39	3	1.26	3	0.75	3

Table 4. Values of the bandwidth h and TRIMMED for the 7 cities for the kernel smoothing estimate, local polynomial smoothing estimate, and spline smoothing estimate of $T0.05(t_j)$.

City	KS_h	KS_Trim	LPS_h	LPS_Trim	SS_h	SS_Trim
DAL	1.17	3	10.01	3	0.35	3
KANSAS	1.24	3	10.01	3	0.26	3
MIAMI	1.11	3	1.01	3	0.59	3
MINNEAPOLIS	1.31	3	10.01	3	0.25	3
PORTLAND	10.01	3	10.01	3	1.51	3
SAN DIEGO	4.05	3	1.55	3	0.70	3
SEATTLE	1.01	3	1.01	3	0.52	3

at the figures, we see that KS and LP estimators are a little rough compared to the SS estimator, and the SS estimator produces narrower bootstrap confidence bands. A close look at Figure 1 tells that there is not much change in Miami for $T_{0.95}(t_j)$. In San Diego and Seattle, we see from Figure 2 that there is an upward trend in $T0.05(t_j)$ from 1990 to 2016. In all figures, we see that two-step smoothing estimators better approximate the extreme quantiles than quantile regression line. Tables 5 to 11 show nonparametric raw quantile values ($T_{0.95}(t_j)$, and $T_{0.05}(t_j)$), two-step kernel smoothing estimates ($KS_{.95}(t_j)$ and $KS_{.05}(t_j)$), two-step local polynomial smoothing estimates ($LP_{.95}(t_j)$ and $LP_{.95}(t_j)$), two-step spline smoothing estimates ($SS_{.95}(t_j)$ and $SS_{.95}(t_j)$), and fitted quantile regression values ($QR_{.95}(t_j)$ and $QR_{.95}(t_j)$) for 95th and 5th percentile values from 1990 to 2016 for each of the 7 cities. Tabular representation of pointwise bootstrap confidence band have been omitted to avoid redundancy. From Tables 5 to 11, we see that all three two-step smoothing values better approximate the values of $T_{.95}(t_j)$ and $T_{.95}(t_j)$ than the values obtained by the quantile regression line in most of the 27 time points.

6. Discussion

We proposed and developed three two-step smoothing estimators for smoothing estimation of time-variant nonparametric extreme quantiles. We compared their performances among themselves and also compared them against quantile regression. We showed by application and simulation that smoothing curves obtained from these smoothing estimators outperformed the quantile regression line in terms of smaller MAD values, narrower bootstrap confidence bands, smaller bias, smaller MSE and higher coverage probability.

There are a number of theoretical and methodological aspects that need to be investigated. Theoretical and simulation studies are needed to investigate the properties of other smoothing methods, such as B-splines, wavelets and other basis approximations, and their corresponding asymptotic inference procedures. If data can be approximated by a parametric probability model, then the one-step kernel log likelihood smoothing method could also be investigated. However, it is extremely hard to approximate time-variant data by a parametric probability model. Under robustness assumptions, one can check the performance of one-step kernel log likelihood estimation method with the above estimation methods.

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Manuscript received 2020-05-01, revised 2021-02-09, accepted 2021-07-01.

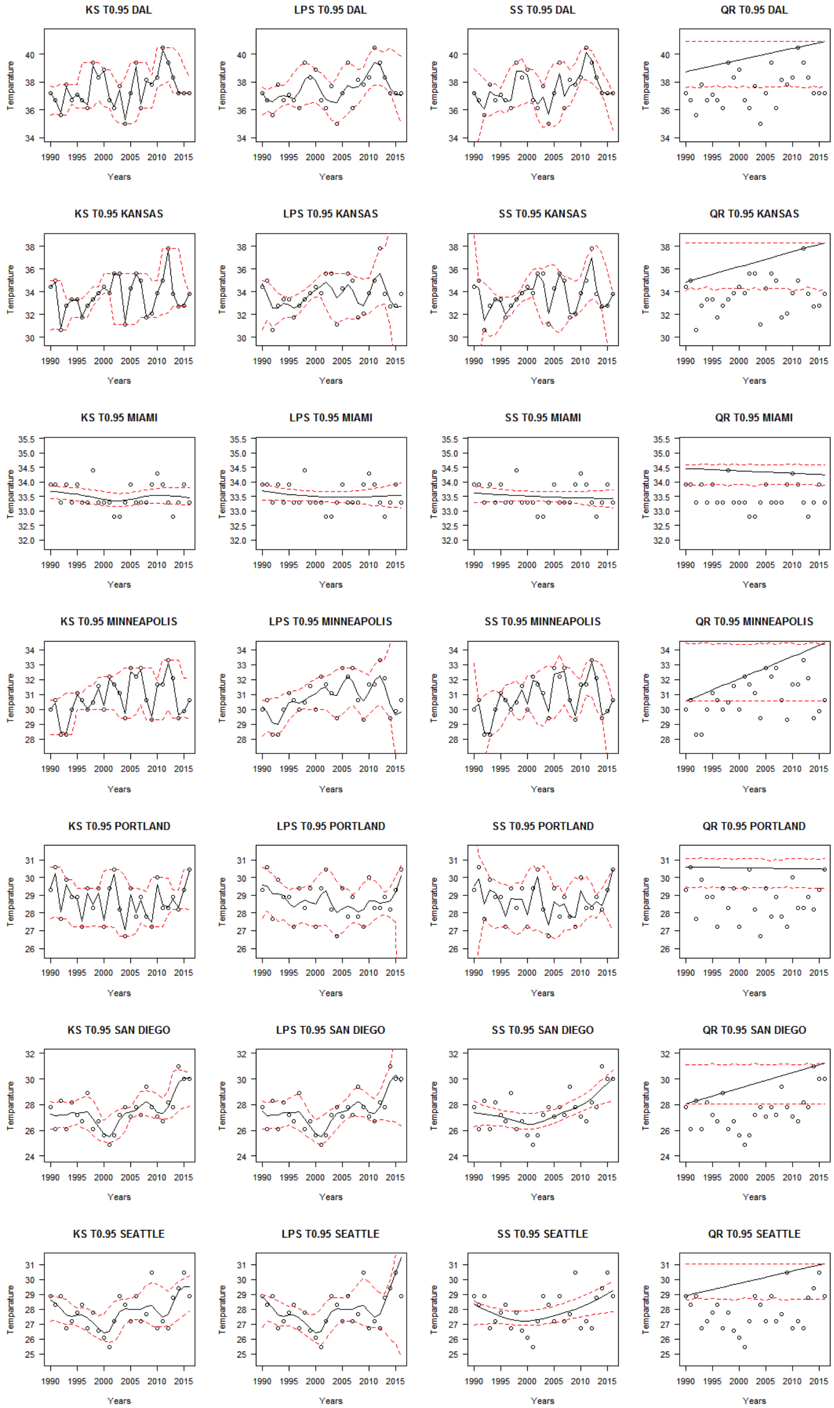

Figure 1. KS, LP, SS and QR estimates (solid lines) of T0.95 together with point-wise bootstrap confidence bands (dashed lines).

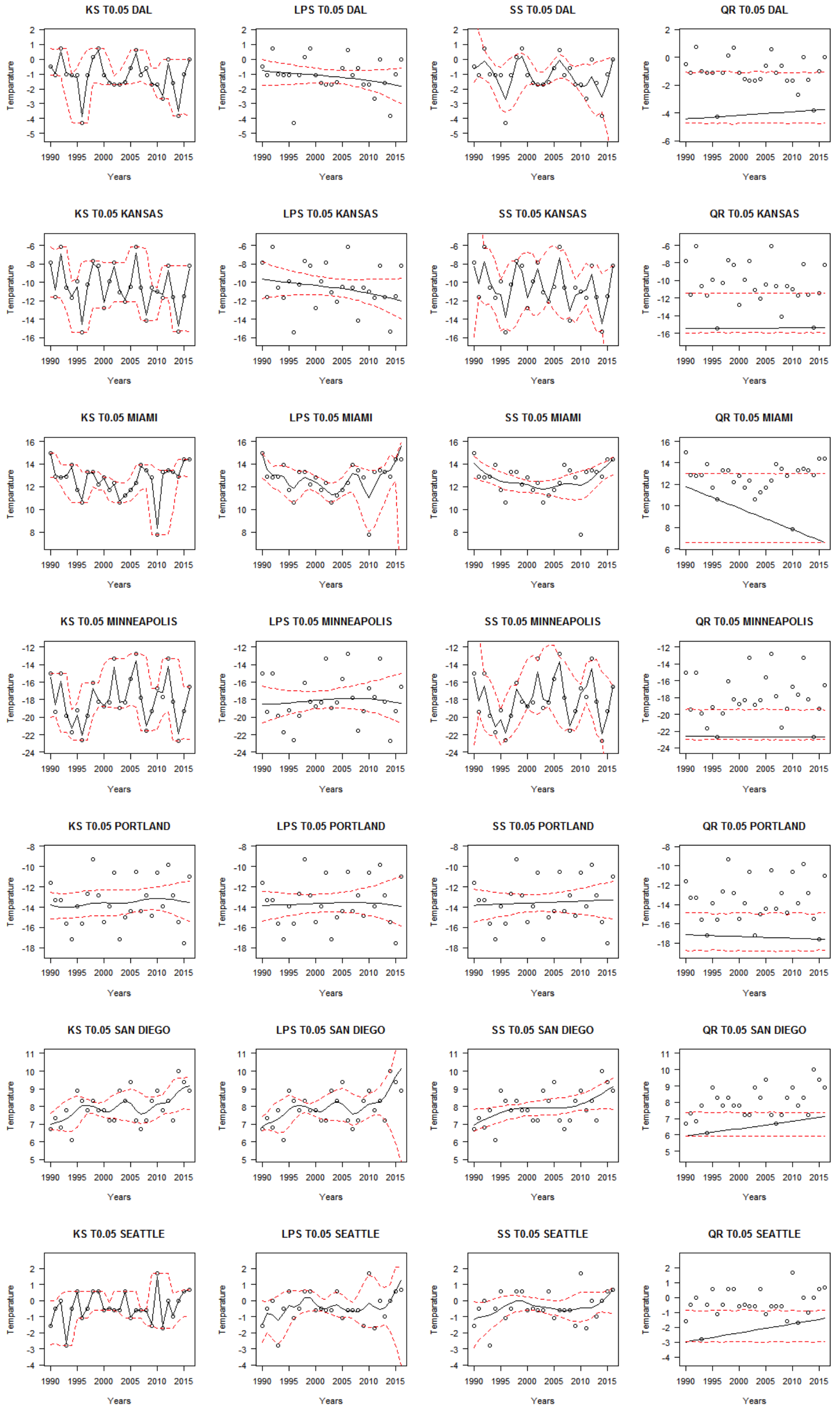

Figure 2. KS, LP, SS and QR estimates (solid lines) of T0.05 together with point-wise bootstrap confidence bands (dashed lines).

Table 5. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from Dallas in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	37.20	37.17	37.21	37.22	38.74	−0.48	−0.52	−0.77	−0.63	−4.43
1991	36.70	36.67	36.66	36.51	38.82	−1.10	−0.95	−0.81	−0.47	−4.41
1992	35.60	35.79	36.56	36.08	38.90	0.73	0.51	−0.85	−0.11	−4.38
1993	37.80	37.61	36.89	37.28	38.98	−1.00	−0.90	−0.88	−0.54	−4.35
1994	36.70	36.78	37.00	37.02	39.07	−1.10	−1.09	−0.91	−1.14	−4.33
1995	37.10	37.05	36.89	36.96	39.15	−1.10	−1.30	−0.94	−1.99	−4.30
1996	36.70	36.69	36.81	36.55	39.23	−4.28	−3.88	−0.97	−2.71	−4.28
1997	36.10	36.32	37.24	36.67	39.32	−1.10	−1.22	−0.99	−1.54	−4.25
1998	39.40	39.15	38.13	38.75	39.40	0.12	0.08	−1.02	−0.11	−4.22
1999	38.30	38.40	38.48	38.76	39.48	0.70	0.55	−1.05	0.19	−4.20
2000	38.90	38.74	38.07	38.50	39.57	−1.10	−1.02	−1.07	−0.69	−4.17
2001	36.70	36.79	37.24	36.90	39.65	−1.58	−1.56	−1.10	−1.47	−4.14
2002	36.10	36.22	36.74	36.42	39.73	−1.70	−1.69	−1.13	−1.78	−4.12
2003	37.68	37.44	36.58	36.93	39.82	−1.70	−1.69	−1.16	−1.73	−4.09
2004	35.00	35.28	36.50	35.69	39.90	−1.55	−1.50	−1.20	−1.31	−4.06
2005	37.20	37.20	37.16	37.22	39.98	−0.60	−0.58	−1.23	−0.53	−4.04
2006	39.40	39.09	37.74	38.62	40.06	0.60	0.42	−1.27	−0.07	−4.01
2007	36.10	36.40	37.56	36.98	40.15	−1.10	−0.96	−1.31	−0.50	−3.98
2008	38.18	38.04	37.66	37.68	40.23	−0.60	−0.70	−1.35	−0.95	−3.96
2009	37.80	37.85	38.08	37.89	40.31	−1.70	−1.63	−1.40	−1.53	−3.93
2010	38.30	38.40	38.75	38.59	40.40	−1.70	−1.76	−1.44	−1.88	−3.91
2011	40.48	40.30	39.39	40.09	40.48	−2.68	−2.46	−1.50	−1.75	−3.88
2012	39.40	39.40	39.19	39.53	40.56	0.00	−0.26	−1.55	−1.18	−3.85
2013	38.30	38.30	38.35	38.27	40.65	−1.60	−1.64	−1.61	−1.83	−3.83
2014	37.20	37.26	37.54	37.30	40.73	−3.80	−3.49	−1.68	−2.55	−3.80
2015	37.20	37.20	37.11	37.15	40.81	−1.00	−1.11	−1.75	−1.55	−3.77
2016	37.20	37.20	37.02	37.20	40.90	0.00	−0.07	−1.82	−0.03	−3.75

Table 6. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from Kansas City in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	34.40	34.42	34.73	34.61	34.87	−7.80	−8.12	−9.64	−8.27	−15.47
1991	35.00	34.86	33.76	34.28	35.00	−11.58	−10.85	−9.72	−10.13	−15.47
1992	30.60	30.79	32.59	31.45	35.13	−6.10	−6.89	−9.80	−7.76	−15.47
1993	32.80	32.75	32.54	32.42	35.27	−10.60	−10.33	−9.87	−9.91	−15.46
1994	33.30	33.29	32.95	33.36	35.40	−11.70	−11.47	−9.94	−11.20	−15.46
1995	33.30	33.25	32.84	33.09	35.53	−9.88	−10.46	−10.01	−11.33	−15.45
1996	31.70	31.78	32.55	32.03	35.67	−15.45	−14.60	−10.07	−13.86	−15.45
1997	32.80	32.78	32.75	32.63	35.80	−10.26	−10.47	−10.13	−10.78	−15.45
1998	33.30	33.30	33.30	33.33	35.93	−7.68	−7.92	−10.19	−7.85	−15.44
1999	33.90	33.90	33.81	33.92	36.07	−8.20	−8.52	−10.26	−8.80	−15.44
2000	34.40	34.37	34.16	34.22	36.20	−12.80	−12.21	−10.32	−11.71	−15.43
2001	33.90	33.96	34.51	34.23	36.33	−9.88	−9.95	−10.39	−10.06	−15.43
2002	35.60	35.55	34.86	35.52	36.47	−7.80	−8.22	−10.46	−8.52	−15.43
2003	35.60	35.47	34.34	34.95	36.60	−11.10	−10.92	−10.53	−10.75	−15.42
2004	31.10	31.32	33.50	32.05	36.73	−12.08	−11.87	−10.61	−11.93	−15.42
2005	34.30	34.25	33.94	33.92	36.87	−10.48	−10.26	−10.69	−9.92	−15.42
2006	35.60	35.55	34.66	35.59	37.00	−6.10	−6.80	−10.77	−7.29	−15.41
2007	35.00	34.92	34.12	34.70	37.13	−10.60	−10.52	−10.87	−10.39	−15.41
2008	31.70	31.81	32.96	32.08	37.27	−14.13	−13.57	−10.96	−13.17	−15.40
2009	32.10	32.14	32.81	32.08	37.40	−10.60	−10.91	−11.07	−11.40	−15.40
2010	33.90	33.88	33.83	33.72	37.53	−11.00	−11.02	−11.18	−11.05	−15.40
2011	35.00	35.05	35.15	35.46	37.67	−11.70	−11.37	−11.30	−10.85	−15.39
2012	37.80	37.61	35.60	37.02	37.80	−8.18	−8.72	−11.43	−9.15	−15.39
2013	33.78	33.86	34.48	34.26	37.93	−11.60	−11.63	−11.56	−11.70	−15.38
2014	32.68	32.71	33.17	32.65	38.07	−15.38	−14.77	−11.71	−14.47	−15.38
2015	32.80	32.83	32.66	32.80	38.20	−11.48	−11.53	−11.86	−11.94	−15.38
2016	33.81	33.78	32.69	33.78	38.33	−8.20	−8.48	−12.02	−8.19	−15.37

Table 7. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from Miami in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	33.90	33.68	33.70	33.61	34.47	15.00	14.90	14.83	14.09	11.80
1991	33.90	33.66	33.67	33.60	34.46	12.90	12.99	13.53	13.64	11.60
1992	33.30	33.64	33.64	33.59	34.45	12.80	12.81	13.03	13.25	11.40
1993	33.90	33.62	33.62	33.58	34.44	12.90	12.94	13.05	12.93	11.20
1994	33.30	33.60	33.59	33.57	34.43	13.90	13.75	12.87	12.68	11.00
1995	33.90	33.57	33.57	33.56	34.43	11.70	11.75	12.14	12.48	10.80
1996	33.30	33.54	33.55	33.55	34.42	10.60	10.78	11.87	12.38	10.60
1997	33.30	33.51	33.54	33.54	34.41	13.30	13.17	12.50	12.36	10.40
1998	34.40	33.47	33.52	33.53	34.40	13.30	13.25	12.84	12.35	10.20
1999	33.30	33.43	33.51	33.52	34.39	12.20	12.28	12.64	12.28	10.00
2000	33.30	33.39	33.50	33.51	34.38	12.80	12.72	12.38	12.17	9.80
2001	33.30	33.36	33.49	33.50	34.38	11.70	11.78	12.09	12.01	9.60
2002	32.80	33.35	33.48	33.50	34.37	12.32	12.21	11.72	11.88	9.40
2003	32.80	33.35	33.48	33.49	34.36	10.60	10.71	11.32	11.81	9.20
2004	33.30	33.36	33.48	33.48	34.35	11.25	11.24	11.34	11.84	9.00
2005	33.90	33.39	33.47	33.48	34.34	11.70	11.71	11.81	11.98	8.80
2006	33.30	33.43	33.47	33.47	34.33	12.32	12.37	12.55	12.14	8.60
2007	33.30	33.47	33.48	33.46	34.33	13.90	13.80	13.18	12.24	8.40
2008	33.30	33.51	33.48	33.46	34.32	13.45	13.44	13.02	12.24	8.20
2009	33.90	33.53	33.48	33.45	34.31	12.80	12.59	11.84	12.16	8.00
2010	34.30	33.54	33.49	33.45	34.30	7.80	8.30	11.03	12.16	7.80
2011	33.90	33.54	33.50	33.44	34.29	13.30	13.05	11.98	12.36	7.60
2012	33.30	33.53	33.51	33.44	34.28	13.45	13.44	13.03	12.69	7.40
2013	32.80	33.51	33.51	33.43	34.28	13.30	13.29	13.28	13.07	7.20
2014	33.30	33.49	33.52	33.43	34.27	12.90	12.99	13.47	13.48	7.00
2015	33.90	33.48	33.53	33.42	34.26	14.40	14.33	14.24	13.93	6.80
2016	33.30	33.46	33.54	33.42	34.25	14.40	14.40	15.58	14.38	6.60

Table 8. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from Minneapolis in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	30.00	30.04	30.21	30.08	30.55	−15.00	−15.47	−18.54	−15.41	−22.63
1991	30.60	30.41	29.82	30.38	30.70	−19.40	−18.55	−18.53	−18.13	−22.64
1992	28.30	28.45	29.07	28.47	30.85	−15.00	−15.90	−18.51	−16.45	−22.64
1993	28.30	28.41	29.01	28.34	31.00	−19.88	−19.58	−18.47	−19.37	−22.64
1994	30.00	29.96	29.80	29.97	31.15	−21.70	−21.28	−18.43	−21.12	−22.64
1995	31.10	31.00	30.48	31.04	31.30	−19.18	−19.76	−18.38	−20.27	−22.65
1996	30.60	30.59	30.53	30.61	31.45	−22.65	−22.05	−18.33	−21.82	−22.65
1997	30.00	30.07	30.41	30.04	31.60	−19.88	−19.78	−18.27	−19.76	−22.65
1998	30.48	30.52	30.62	30.57	31.75	−16.10	−16.67	−18.21	−16.82	−22.66
1999	31.58	31.41	30.87	31.29	31.90	−18.20	−18.05	−18.15	−17.87	−22.66
2000	30.00	30.24	31.04	30.38	32.05	−18.75	−18.65	−18.09	−18.82	−22.66
2001	32.20	32.03	31.43	31.93	32.20	−18.30	−17.86	−18.04	−17.48	−22.66
2002	31.70	31.69	31.46	31.82	32.35	−13.30	−14.33	−17.99	−14.81	−22.67
2003	31.10	31.03	30.99	30.87	32.50	−18.90	−18.30	−17.95	−17.97	−22.67
2004	29.40	29.73	30.95	29.85	32.65	−18.30	−18.10	−17.91	−18.27	−22.67
2005	32.80	32.54	31.72	32.37	32.80	−15.60	−15.59	−17.88	−15.38	−22.68
2006	32.20	32.28	32.23	32.49	32.95	−12.80	−13.55	−17.86	−13.64	−22.68
2007	32.80	32.62	31.91	32.58	33.10	−17.80	−17.68	−17.86	−17.60	−22.68
2008	30.60	30.66	30.98	30.62	33.25	−21.55	−20.97	−17.86	−20.98	−22.68
2009	29.30	29.54	30.56	29.55	33.40	−19.30	−19.27	−17.88	−19.45	−22.69
2010	31.70	31.55	31.15	31.39	33.55	−16.70	−17.05	−17.91	−17.28	−22.69
2011	31.70	31.80	31.96	31.96	33.70	−17.68	−17.16	−17.95	−16.57	−22.69
2012	33.30	33.12	32.28	33.14	33.85	−13.30	−14.20	−18.01	−14.46	−22.69
2013	32.10	32.00	31.57	32.00	34.00	−18.20	−18.16	−18.09	−18.13	−22.70
2014	29.40	29.60	30.30	29.62	34.15	−22.70	−21.94	−18.18	−21.88	−22.70
2015	29.88	29.90	29.63	29.79	34.30	−19.30	−19.36	−18.29	−19.75	−22.70
2016	30.60	30.55	29.84	30.60	34.45	−16.51	−16.81	−18.42	−16.49	−22.71

Table 9. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from Portland in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	29.30	29.42	29.59	29.51	30.60	−11.58	−13.79	−13.81	−13.81	−17.13
1991	30.60	30.24	29.52	29.92	30.60	−13.30	−13.89	−13.80	−13.79	−17.15
1992	27.65	28.09	29.08	28.53	30.60	−13.30	−13.96	−13.80	−13.77	−17.16
1993	29.88	29.61	29.07	29.31	30.59	−15.60	−13.99	−13.79	−13.75	−17.18
1994	28.90	28.98	28.98	29.15	30.59	−17.20	−13.97	−13.77	−13.73	−17.20
1995	28.90	28.76	28.56	28.56	30.58	−13.90	−13.90	−13.75	−13.70	−17.22
1996	27.20	27.53	28.32	27.81	30.58	−15.60	−13.81	−13.73	−13.68	−17.24
1997	29.40	29.12	28.56	28.82	30.57	−12.68	−13.71	−13.71	−13.66	−17.25
1998	28.30	28.48	28.72	28.77	30.57	−9.30	−13.63	−13.69	−13.64	−17.27
1999	29.40	29.12	28.60	28.78	30.56	−12.80	−13.58	−13.67	−13.62	−17.29
2000	27.20	27.57	28.52	27.88	30.56	−15.45	−13.57	−13.64	−13.60	−17.31
2001	29.40	29.31	29.04	29.26	30.55	−13.90	−13.58	−13.62	−13.58	−17.33
2002	30.48	30.20	29.26	30.06	30.55	−10.60	−13.60	−13.60	−13.56	−17.34
2003	28.20	28.27	28.52	28.30	30.54	−17.20	−13.61	−13.58	−13.54	−17.36
2004	26.70	27.05	28.01	27.30	30.54	−15.00	−13.58	−13.57	−13.52	−17.38
2005	29.40	29.04	28.26	28.63	30.53	−14.40	−13.53	−13.56	−13.50	−17.40
2006	27.80	28.03	28.39	28.36	30.53	−10.48	−13.44	−13.55	−13.48	−17.42
2007	28.90	28.72	28.30	28.53	30.52	−14.40	−13.34	−13.55	−13.46	−17.44
2008	27.80	27.84	28.05	27.81	30.52	−12.80	−13.24	−13.55	−13.44	−17.45
2009	27.20	27.49	28.19	27.75	30.51	−14.88	−13.17	−13.56	−13.42	−17.47
2010	30.00	29.62	28.68	29.28	30.51	−10.60	−13.14	−13.58	−13.40	−17.49
2011	28.30	28.44	28.68	28.65	30.50	−13.90	−13.14	−13.61	−13.38	−17.51
2012	28.30	28.35	28.53	28.37	30.50	−9.85	−13.19	−13.64	−13.36	−17.53
2013	28.90	28.79	28.58	28.63	30.49	−12.80	−13.26	−13.69	−13.34	−17.54
2014	28.20	28.35	28.70	28.44	30.49	−15.50	−13.35	−13.75	−13.32	−17.56
2015	29.30	29.31	29.17	29.23	30.48	−17.58	−13.44	−13.83	−13.30	−17.58
2016	30.48	30.37	30.11	30.47	30.48	−11.00	−13.53	−13.92	−13.27	−17.60

Table 10. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from San Diago in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	27.80	27.24	27.51	27.40	28.05	6.70	6.99	6.83	6.94	5.92
1991	26.10	27.12	27.12	27.34	28.18	7.32	7.06	7.01	7.10	5.96
1992	28.30	27.19	27.19	27.29	28.30	6.83	7.15	7.13	7.25	6.01
1993	26.10	27.22	27.22	27.22	28.42	7.80	7.27	7.27	7.39	6.05
1994	28.20	27.36	27.36	27.15	28.55	6.10	7.51	7.51	7.53	6.10
1995	27.20	27.36	27.36	27.06	28.67	8.90	7.81	7.81	7.65	6.15
1996	26.70	27.40	27.40	26.95	28.79	8.30	8.03	8.01	7.75	6.19
1997	28.90	27.43	27.43	26.83	28.91	7.80	8.07	8.06	7.82	6.24
1998	26.10	26.93	26.93	26.69	29.04	8.30	7.99	7.98	7.87	6.28
1999	26.70	26.31	26.31	26.58	29.16	7.80	7.85	7.85	7.89	6.33
2000	25.60	25.75	25.75	26.50	29.28	7.80	7.71	7.72	7.91	6.38
2001	24.88	25.50	25.50	26.50	29.40	7.20	7.68	7.70	7.91	6.42
2002	25.60	25.96	25.96	26.57	29.53	7.20	7.86	7.87	7.92	6.47
2003	27.20	26.81	26.81	26.70	29.65	8.90	8.15	8.14	7.93	6.52
2004	27.80	27.33	27.33	26.88	29.77	8.30	8.31	8.29	7.93	6.56
2005	27.10	27.45	27.45	27.06	29.90	9.40	8.16	8.14	7.92	6.61
2006	27.80	27.58	27.58	27.24	30.02	7.20	7.80	7.81	7.91	6.65
2007	27.20	27.91	27.91	27.42	30.14	6.70	7.55	7.57	7.91	6.70
2008	29.40	28.22	28.22	27.60	30.26	7.20	7.63	7.65	7.95	6.75
2009	27.80	27.92	27.92	27.77	30.39	8.30	7.93	7.92	8.02	6.79
2010	27.10	27.38	27.38	27.96	30.51	8.90	8.14	8.13	8.12	6.84
2011	26.70	27.33	27.33	28.20	30.63	7.80	8.19	8.19	8.24	6.88
2012	28.18	27.84	27.84	28.50	30.75	8.30	8.25	8.26	8.38	6.93
2013	27.80	28.74	28.75	28.86	30.88	7.20	8.48	8.53	8.55	6.98
2014	31.00	29.72	29.85	29.26	31.00	10.00	8.83	9.07	8.73	7.02
2015	30.00	30.11	30.33	29.66	31.12	9.40	9.08	9.70	8.92	7.07
2016	30.00	30.07	29.76	30.06	31.25	8.90	9.15	10.15	9.10	7.12

Table 11. Raw estimates ($T_{.95}(t_j)$, $T_{.05}(t_j)$), kernel smoothing estimates ($KS_{.95}(t_j)$, $KS_{.05}(t_j)$), local polynomial smoothing estimates ($LP_{.95}(t_j)$, $LP_{.05}(t_j)$), spline smoothing estimate ($SS_{.95}(t_j)$, $SS_{.05}(t_j)$), and quantile regression estimate ($QR_{.95}(t_j)$, $QR_{.05}(t_j)$) of the 95th and 5th percentile temperature from Seattle in the United States from 1990 to 2016.

t_j	$T_{.95}(t_j)$	$KS_{.95}(t_j)$	$LP_{.95}(t_j)$	$SS_{.95}(t_j)$	$QR_{.95}(t_j)$	$T_{.05}(t_j)$	$KS_{.05}(t_j)$	$LP_{.05}(t_j)$	$SS_{.05}(t_j)$	$QR_{.05}(t_j)$
1990	28.90	28.62	28.85	28.34	28.90	−1.58	−1.55	−1.48	−1.18	−2.98
1991	28.30	28.43	28.54	28.16	28.98	−0.48	−0.50	−0.81	−1.03	−2.92
1992	28.90	28.06	28.09	27.99	29.07	0.00	−0.09	−0.90	−0.96	−2.86
1993	26.70	27.64	27.64	27.82	29.15	−2.80	−2.66	−1.23	−0.91	−2.80
1994	27.20	27.49	27.49	27.67	29.23	−0.48	−0.51	−0.79	−0.72	−2.74
1995	27.80	27.60	27.60	27.53	29.32	0.60	0.53	−0.32	−0.49	−2.68
1996	28.30	27.61	27.61	27.41	29.40	−1.10	−1.04	−0.41	−0.32	−2.62
1997	26.70	27.41	27.41	27.32	29.48	−0.48	−0.47	−0.25	−0.14	−2.56
1998	27.80	27.11	27.11	27.24	29.57	0.60	0.57	0.17	−0.01	−2.49
1999	26.58	26.71	26.71	27.20	29.65	0.60	0.57	0.18	−0.01	−2.43
2000	26.10	26.39	26.39	27.20	29.73	−0.60	−0.56	−0.21	−0.14	−2.37
2001	25.48	26.53	26.53	27.23	29.81	−0.48	−0.49	−0.47	−0.28	−2.31
2002	27.20	27.19	27.19	27.29	29.90	−0.60	−0.60	−0.50	−0.36	−2.25
2003	28.90	27.85	27.85	27.38	29.98	−0.60	−0.57	−0.33	−0.39	−2.19
2004	28.30	28.06	28.06	27.47	30.06	0.60	0.52	−0.25	−0.43	−2.13
2005	27.20	28.01	28.01	27.57	30.15	−1.10	−1.04	−0.51	−0.52	−2.07
2006	28.90	27.97	27.97	27.68	30.23	−0.60	−0.61	−0.66	−0.59	−2.01
2007	27.20	28.01	28.01	27.78	30.31	−0.60	−0.60	−0.67	−0.62	−1.94
2008	27.65	28.21	28.21	27.89	30.40	−0.60	−0.63	−0.72	−0.61	−1.88
2009	30.48	28.26	28.25	28.01	30.48	−1.58	−1.47	−0.49	−0.54	−1.82
2010	26.70	27.83	27.83	28.13	30.56	1.70	1.52	−0.16	−0.45	−1.76
2011	27.20	27.46	27.46	28.28	30.65	−1.70	−1.56	−0.41	−0.47	−1.70
2012	26.70	27.68	27.69	28.44	30.73	0.00	−0.07	−0.56	−0.43	−1.64
2013	28.78	28.42	28.47	28.63	30.81	−1.00	−0.95	−0.45	−0.30	−1.58
2014	29.40	29.17	29.47	28.83	30.90	0.00	−0.01	−0.04	−0.04	−1.52
2015	30.48	29.53	30.47	29.03	30.98	0.60	0.59	0.61	0.30	−1.46
2016	28.90	29.51	31.46	29.23	31.06	0.68	0.67	1.25	0.66	−1.39